# Math 19: Fall 2014
# Midterm 1
Blue version

**NAME:** SOLUTIONS

**LECTURE:**

Time: **75 minutes**

For each problem, you should write down all of your work carefully and legibly to receive full credit. When asked to justify your answer, you should use theorems and/or mathematical reasoning to support your answer, as appropriate.

Failure to follow these instructions will constitute a breach of the Stanford Honor Code:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason.
- You are required to sit in your assigned seat.
- You are bound by the Stanford Honor Code, which stipulates among other things that you may not communicate with anyone other than the instructor during the exam, or look at anyone else's solutions.

I understand and accept these instructions.

Signature: ______________________________

| Problem | Value | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 5 | |
| 4 | 5 | |
| 5 | 28 | |
| 6 | 8 | |
| 7 | 14 | |
| 8 | 11 | |
| 9 | 9 | |
| TOTAL | 100 | |

**Problem 1 : (10 points)**

a) (4 points) Suppose that $f$ is a function that is one-to-one. What is the definition of the inverse of $f$?

If $f$ has domain $A$ and range $B$, then $f^{-1}$ has domain $B$ and range $A$, and is defined by

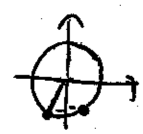
$$f^{-1}(y) = x \text{ if and only if } f(x) = y$$

for any $y$ in $B$.

b) (4 points) Consider the function $f\colon [-1, 1] \to \mathbb{R}$, $f(x) = \arcsin x$. What **function** is the inverse of $f$?

The function $f^{-1}\colon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$

$$f^{-1}(x) = \sin x$$



c) (2 points) Find the value:

$$\arcsin\left(\sin\left(\frac{7\pi}{6}\right)\right).$$

Blue version:

$$\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$$

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

Green version

$$\arcsin\left(\sin\left(\frac{4\pi}{3}\right)\right)$$

$$\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

**Problem 2 : (10 points)**

a) Simplify the following number completely:

$$\sin\left(\frac{5\pi}{12}\right)$$

Hint: $\frac{5}{12} = \frac{1}{4} + \frac{1}{6}$

$$\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{6} + \cos\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Green version:

$$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

b) Simplify the following number completely:

$$\log_2 8 - \log_2 \frac{1}{4}$$

$$\log_2 8 - \log_2 \frac{1}{4} = \log_2 2^3 - \log_2 2^{-2} = 3-(-2) = 5$$

Green version:

$$\log_3 \frac{1}{27} - \log_3 9 = \log_3 3^{-3} - \log_3 3^2 = -3-2 = -5$$

**Problem 3 : (5 points)** Give the form of the partial fraction decomposition. You do not need to solve for the constants.

$$\frac{2x^2+x+1}{x^3-x^2+x-1}$$

Factor denominator:

$$x^3-x^2+x-1 = x^2(x-1)+1(x-1) = \underbrace{(x-1)}_{\text{linear}}\underbrace{(x^2+1)}_{\text{irred. quadratic}}$$

Form of the decomposition:

$$\frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

Green version:

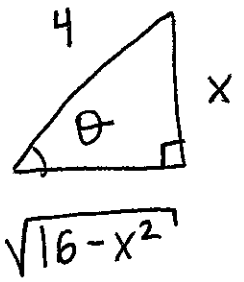
Factor denominator:

$$x^3-2x^2+2x-4 = x^2(x-2)+2(x-2) = \underbrace{(x-2)}_{\text{linear}}\underbrace{(x^2+2)}_{\text{irred. quad.}}$$

Form of the decomposition:

$$\frac{A}{x-2} + \frac{Bx+C}{x^2+2}$$

**Problem 4 : (5 points)** Let $\theta$ be angle such that $\sin\theta = \frac{x}{4}$. Write down an expression for $\tan\theta$ in terms of $x$.

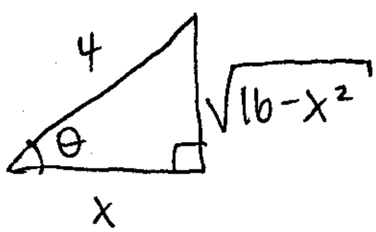



$$\tan\theta = \frac{x}{\sqrt{16-x^2}}$$

↑ found using $a^2+b^2=c^2$ $\quad b^2 = 16-x^2$

$x^2+b^2=16 \quad b=\sqrt{16-x^2}$ (positive square root since $b>0$)

Green version:




← found as in Blue version solution

$$\tan\theta = \frac{\sqrt{16-x^2}}{x}$$

**Problem 5 : (28 points)** Compute the following limits, and answer the follow-up question.

a) i. $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$

plug in: $\frac{0}{0}$ I will multiply top & bottom by conjugate

$$= \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} \frac{(\sqrt{x}+1)}{(\sqrt{x}+1)}$$

$$= \lim_{x\to 1} \frac{x-1}{(x-1)(\sqrt{x}+1)}$$

$$= \lim_{x\to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{2}$$

ii. Let $f$ be a function given by the rule $f(x) = \frac{\sqrt{x}-1}{x-1}$. What is the behavior of $f$ near $x = 1$? Is $f$ continuous? If not, does $f$ have a removable discontinuity, a jump discontinuity, or an infinite discontinuity? In the case where $f$ has a jump discontinuity or an infinite discontinuity, compute the one-sided limits of $f$ on both sides of the discontinuity.

f is not continuous at $x=1$, it has a removable discontinuity.

b) i. $\lim_{x \to -1} \dfrac{x^2 + 3x + 2}{|x+2|}$

plug in: $\dfrac{(-1)^2 + 3(-1) + 2}{|(-1)+2|} = \dfrac{1-3+2}{1} = 0$

$= 0$

ii. Let $f$ be a function given by the rule $f(x) = \frac{x^2+3x+2}{|x+2|}$. What is the behavior of $f$ near $x = -1$? Is $f$ continuous? If not, does $f$ have a removable discontinuity, a jump discontinuity, or an infinite discontinuity? In the case where $f$ has a jump discontinuity or an infinite discontinuity, compute the one-sided limits of $f$ on both sides of the discontinuity.

f is continuous at $x = -1$.

c) i. $\lim_{x \to -2} \frac{x^2+3x+2}{|x+2|}$

plug in: $\frac{(-2)^2+3(-2)+2}{|(-2)+2|} = \frac{4-6+2}{0} = \frac{0}{0}$

I will factor the numerator.

$$\lim_{x \to -2^+} \frac{(x+2)(x+1)}{|x+2|} = \lim_{x \to -2^+} \frac{(x+2)(x+1)}{(x+2)} = \lim_{x \to -2^+} (x+1) = -1$$

$$\lim_{x \to -2^-} \frac{(x+2)(x+1)}{|x+2|} = \lim_{x \to -2^-} \frac{(x+2)(x+1)}{-(x+2)} = \lim_{x \to -2^-} -(x+1) = 1$$

ii. Let $f$ be a function given by the rule $f(x) = \frac{x^2+3x+2}{|x+2|}$. What is the behavior of $f$ near $x = -2$? Is $f$ continuous? If not, does $f$ have a removable discontinuity, a jump discontinuity, or an infinite discontinuity? In the case where $f$ has a jump discontinuity or an infinite discontinuity, compute the one-sided limits of $f$ on both sides of the discontinuity.

f is not continuous at $x=-2$, it has a jump discontinuity.

$\lim_{x \to -2^+} f(x) = -1$, $\lim_{x \to -2^-} f(x) = 1$

d) i. $\lim_{x\to 2} \frac{x^2+3x+2}{x^2-4x+4}$

plug in: $\frac{2^2+3\cdot 2+2}{2^2-4\cdot 2+4} = \frac{4+6+2}{4-8+4} = \frac{12}{0}$

vertical asymptote

$$\lim_{x\to 2^+} \frac{(x+2)(x+1)}{(x-2)^2} = \frac{12}{(0^+)^2} = \frac{12}{0^+} = +\infty$$

$$\lim_{x\to 2^-} \frac{(x+2)(x+1)}{(x-2)^2} = \frac{12}{(0^-)^2} = \frac{12}{0^+} = +\infty$$

ii. Let $f$ be a function given by the rule $f(x) = \frac{x^2+3x+2}{x^2-4x+4}$. What is the behavior of $f$ near $x = 2$? Is $f$ continuous? If not, does $f$ have a removable discontinuity, a jump discontinuity, or an infinite discontinuity? In the case where $f$ has a jump discontinuity or an infinite discontinuity compute the one-sided limits of $f$ on both sides of the discontinuity.

f is not continuous at $x=2$, f has a vertical asymptote,

$\lim_{x\to 2^+} f(x) = +\infty$, $\lim_{x\to 2^-} f(x) = +\infty$

**Problem 6 : (8 points)** Let $f$ be a function given by the rule

$$f(x) = \frac{x+2}{x+5}.$$

a) Give the rule for the function $f^{-1}$.

$y = \frac{x+2}{x+5}$ $\rightsquigarrow$ flip x and y: $x = \frac{y+2}{y+5}$

Solve for y:

$$x(y+5) = y+2$$
$$xy + 5x = y+2$$
$$xy - y = -5x+2$$
$$y(x-1) = -5x+2$$

$$\boxed{y = \frac{-5x+2}{x-1}}$$

b) What is the domain of $f^{-1}$? Give your answer in interval notation.

The domain is $(-\infty, 1) \cup (1, \infty)$

(all but $x=1$, which makes denominator 0)

c) What is the range of $f$? Give your answer in interval notation.

The range of f is the domain of $f^{-1}$!

$(-\infty, 1) \cup (1, \infty)$

NOTE: It is otherwise very hard to find range!

**Problem 7 : (14 points)** Consider the function $f$ given by the rule

$$f(x) = \frac{3x^2 + 9x + 6}{x^2 + 3x + 2}.$$

a) (3 points) Find all the $x$- and $y$-intercepts of $f$.

y-intercept: $x=0$ $\quad y = \frac{3\cdot 0^2 + 9\cdot 0 + 6}{0^2 + 3\cdot 0 + 2} = \frac{6}{2} = 3$ $\quad (0,3)$

x-intercept: fraction=0 i.e. top=0, bottom≠0

$y = \frac{3(x^2+3x+2)}{(x+2)(x+1)} = \frac{3(x+2)(x+1)}{(x+2)(x+1)}$ $\quad$ no x-intercept

b) (4 points) Use limits to find the equation(s) for all horizontal asymptote(s) of $f$.

$$\lim_{x\to\infty} \frac{3x^2+9x+6}{x^2+3x+2} = \lim_{x\to\infty} \frac{x^2\left(3+\frac{9}{x}+\frac{6}{x^2}\right)}{x^2\left(1+\frac{3}{x}+\frac{2}{x^2}\right)} = 3$$

$$\lim_{x\to-\infty} \frac{3x^2+9x+6}{x^2+3x+2} = \lim_{x\to-\infty} \frac{x^2\left(3+\frac{9}{x}+\frac{6}{x^2}\right)}{x^2\left(1+\frac{3}{x}+\frac{2}{x^2}\right)} = 3$$

f has horizontal asymptote $y=3$ on the left and on the right.

c) (3 points) Find the equation(s) for all vertical asymptote(s) of $f$.

v.a. is when bottom=0, top≠0 } no vertical asymptote

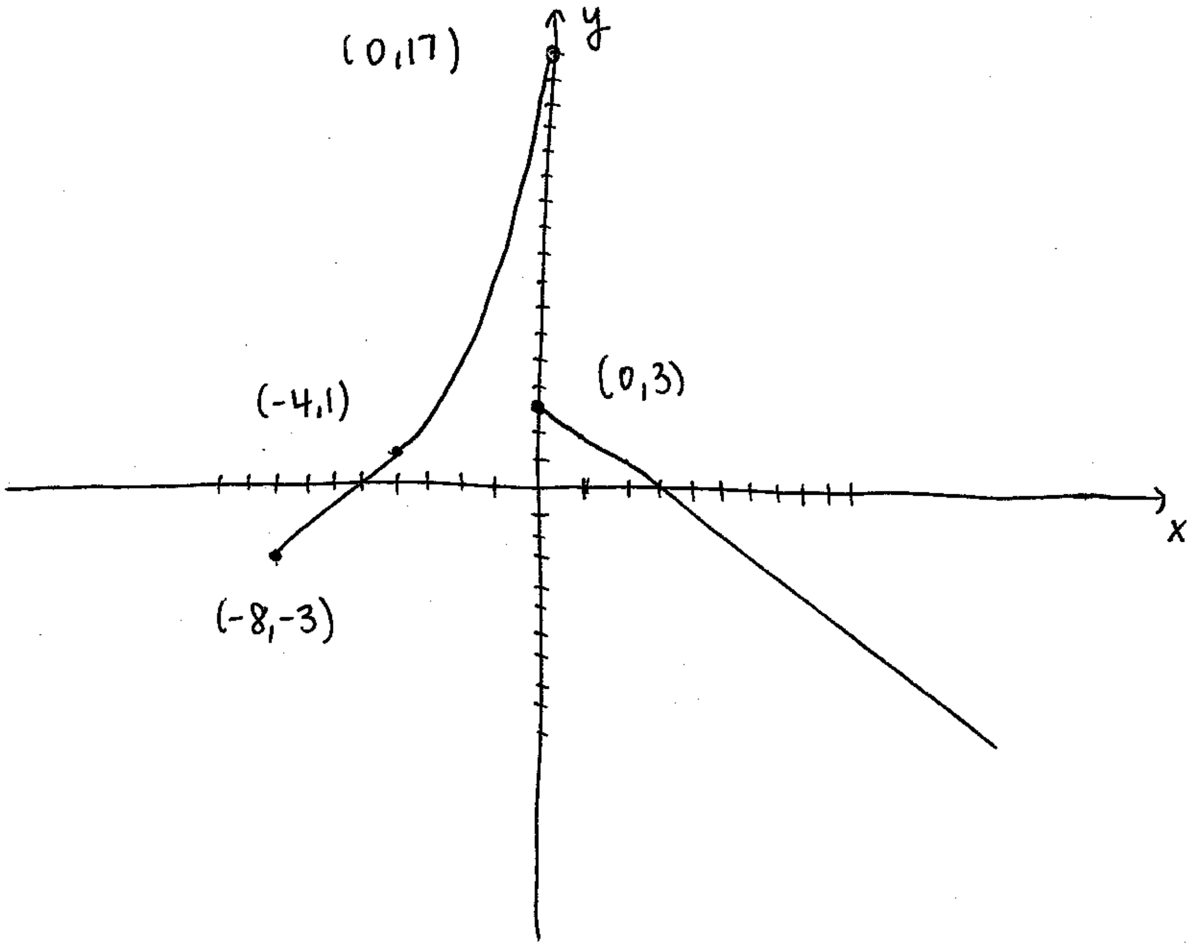
d) (4 points) For each vertical asymptote, find out the behavior of the function on either side of the asymptote.

no vertical asymptote.

**Problem 8 : (11 points)** Consider the function $f\colon [-8,\infty) \to \mathbb{R}$ given by the rule

$$f(x) = \begin{cases} x+5 & \text{if } -8 \le x \le -4, \\ (x+4)^2+1 & \text{if } -4 < x < 0, \\ 3-x & \text{if } 0 \le x. \end{cases}$$

a) (5 points) Sketch the graph of $y = f(x)$.




b) (2 points) Is $f$ one-to-one? Justify.

NO, f does not pass the horizontal line test.

Recall the function we are considering in this problem: $f\colon [-8,\infty) \to \mathbb{R}$,

$$f(x) = \begin{cases} x+5 & \text{if } -8 \le x \le -4, \\ (x+4)^2+1 & \text{if } -4 < x < 0, \\ 3-x & \text{if } 0 \le x. \end{cases}$$

c) (4 points) Is $f$ continuous at $x = -4$? Justify.

$f(-4) = (-4)+5 = 1$ exists

$$\left.\begin{aligned} \lim_{x\to -4^+} f(x) &= \lim_{x\to -4^+} \left[(x+4)^2+1\right] = 1 \\ \lim_{x\to -4^-} f(x) &= \lim_{x\to -4^-} \left[x+5\right] = 1 \end{aligned}\right\} \lim_{x\to -4} f(x) = 1 \text{ exists}$$

$$f(-4) = \lim_{x\to -4} f(x)$$ So $f$ is continuous at $x=-4$.

Green version:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left[3-x\right] = 3$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \left[(x+4)^2+1\right] = 16+1 = 17$$

$\lim_{x\to 0} f(x)$ does not exist, $f$ is not continuous at $x=0$.

**Problem 9 : (9 points)**

a) (5 points) Prove that the polynomial $f(x) = x^5 - 4x + 2$ has a root in the interval $[-1, 1]$.

$f(-1) = (-1)^5 - 4(-1) + 2 = -1 + 4 + 2 = 5$

$f(1) = 1 - 4 \cdot 1 + 2 = 1 - 4 + 2 = -1$

f is positive at $x=-1$ and negative at $x=1$, and f is continuous between $-1$ and $1$ (in fact f is continuous everywhere since it is a polynomial). By the Intermediate Value Theorem, f has a root between $-1$ and $1$.

b) (4 points) Can the same be said of the rational expression $g(x) = \dfrac{x-2}{x}$ ?

No. We cannot apply the Intermediate Value Theorem since g has a discontinuity at $x=0$.

In fact, even though $g(-1) = 3 > 0$ and $g(1) = -1 < 0$, g does not have a zero between $-1$ and $1$. g has only one root and it is at $x=2$.