Midterm 1

Thursday, 10/14/2010

- Complete the following problems. You may use any result from class you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- This is a closed-book exam. No calculators or other electronic aids will be permitted. You are allowed to have one 8.5 by 11 inch sheet of paper containing handwritten notes on both sides.
- In order to receive full credit, *you must show all of your work and justify your answers.* Your answer should be clearly labeled.
- If you need extra room, use the back sides of each page. Staple any scratch paper to your exam.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Full Name: Key.
Circle your lecture time: 9:30 / 11:00

Signature: ____________________

Please leave the following table blank for the grader.

1. ________ (/10 points)
2. ________ (/35 points)
3. ________ (/5 points)
4. ________ (/10 points)
5. ________ (/10 points)
6. ________ (/10 points)

Total. ________ (/80 points)

1. (10 points) Consider the function $y = f(x) = \frac{2x+1}{1-x}$.

(a) (2 points) What is the domain of f? Express your answer as an interval.

$$y = \frac{2x+1}{1-x}$$

must exclude $x = 1$

So $(-\infty, 1) \cup (1, \infty)$

(b) (4 points) Compute f^{-1}, the inverse of f.

Swap x & y:

$$x = \frac{2y+1}{1-y}$$

Solve for y:

$$(1-y)x = 2y+1$$

$$x - xy = 2y+1$$

$$x - 1 = 2y + xy$$

$$x - 1 = (2+x)y$$

$$\boxed{y = \frac{x-1}{2+x}.}$$

(c) (2 points) What is the range of f? Express your answer as an interval.

recall range f = domain f^{-1}

from (b) $f^{-1}(x) = \frac{x-1}{2+x}$

domain f^{-1} : exclude $x = -2$

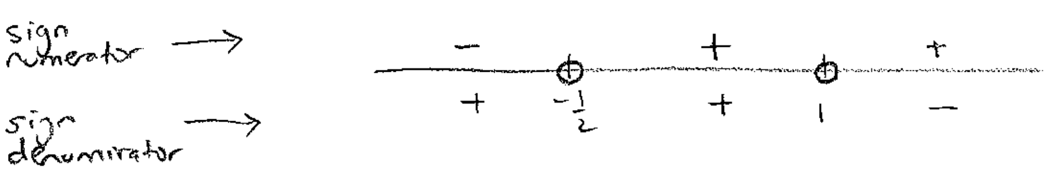
so $(-\infty, -2) \cup (-2, \infty)$

(d) (2 points) What is the domain of

$$\ln\left(\frac{2x+1}{1-x}\right)?$$

Express your answer as an interval.

must require $\frac{2x+1}{1-x} > 0$ for ln to be defined. Let's draw a number line, identifying the zeros of the numerator & denominator



so, the total fraction is positive on $(-\frac{1}{2}, 1)$

2. (35 points) Compute the limit algebraically. No credit will be given for numerical tables. You must show your work at a level appropriate for the particular problem at hand. Do *not* use L'Hopital's rule, if you know it (and definitely do not use it if you don't know it). If a limit does not exist, show why.

(a)

$$\lim_{x\to -1} \frac{4x^2+4x}{x^2-3x-4}.$$

(of type $\frac{0}{0}$, factor)

$4x^2+4x = 4x(x+1)$

$x^2-3x-4 = (x+1)(x-4)$

so $\lim\limits_{x\to -1} \frac{4x^2+4x}{x^2-3x-4} = \lim\limits_{x\to -1} \frac{4x}{x-4} = \frac{-4}{-1-4} = \frac{-4}{-5} = \frac{4}{5}$

(b)

$$\lim_{x\to 1} \frac{x-1}{x^3-1}.$$

(of type $\frac{0}{0}$, factor)

because $p(x) = x^3-1$ satisfies $p(1)=0$ we can factor

$x^3-1 = (x-1)(\,?\,)$

use polynomial division for $(?)$

$$\begin{array}{r|l} & x^2+x+1 \\ x-1 & x^3+0x^2+0x-1 \\ & -(x^3-x^2) \\ & \quad x^2+0x \\ & -(x^2-x) \\ & \quad x-1 \\ & -(x-1) \\ & \quad 0 \end{array}$$

so $x^3-1 = (x-1)(x^2+x+1)$

so, $\frac{x-1}{x^3-1} = \frac{\cancel{(x-1)}}{\cancel{(x-1)}(x^2+x+1)} = \frac{1}{x^2+x+1}$

Finally

$\lim\limits_{x\to 1} \frac{x-1}{x^3-1} = \lim\limits_{x\to 1} \frac{1}{x^2+x+1} = \frac{1}{1+1+1} = \frac{1}{3}.$

(c)

$$\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right). \qquad \left(\text{of type } \infty-\infty\right)$$

$$= \lim_{t\to 0}\left(\frac{t^2+t \quad - \quad t}{t(t^2+t)}\right)$$

$$= \lim_{t\to 0}\left(\frac{t^2}{t(t^2+t)}\right)$$

$$= \lim_{t\to 0}\frac{t}{t^2+t} \qquad \left(\text{of type } \tfrac{0}{0}\right)$$

$$= \lim_{t\to 0}\frac{1}{t+1}$$

$$= 1$$

(d)

$$\lim_{x\to 1} e^{2x}-\ln(x)+x^2+2x+1+\sin(\pi x).$$

all of these functions are continuous at $x=1$

so,

$$\lim_{x\to 1} e^{2x} - \ln(x) + x^2 + 2x + 1 + \sin(\pi x)$$

$$= e^2 - \ln(1) + 1^2 + 2(1) + 1 + \sin(\pi)$$

$$= e^2 - 0 + 4 + 0$$

$$= e^2 + 4$$

(e)

$$\lim_{h\to 0} \frac{1-\sqrt{1+h}}{h}. \qquad \left(\text{of the form } \frac{0}{0}\right)$$

Rationalize!

$$\frac{1-\sqrt{1+h}}{h} = \left(\frac{1-\sqrt{1+h}}{h}\right)\cdot\frac{(1+\sqrt{1+h})}{(1+\sqrt{1+h})} = \frac{1-(1+h)}{h(1+\sqrt{1+h})}$$

← these brackets are key.

$$= \frac{1-1-h}{h(1+\sqrt{1+h})} = \frac{-h}{h(1+\sqrt{1+h})} = \frac{-1}{1+\sqrt{1+h}}.$$

So $\lim_{h\to 0} \frac{1-\sqrt{1+h}}{h} = \lim_{h\to 0} \frac{-1}{1+\sqrt{1+h}} = \frac{-1}{1+\sqrt{1}} = \frac{-1}{1+1} = \frac{-1}{2}.$

(f)

$$\lim_{x\to 0} x^4 \sin\left(\frac{\pi}{x}\right).$$

recall $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$ DNE.

Let's try to squeeze the limit.

Remember $-1 \le \sin\left(\frac{\pi}{x}\right) \le 1$ for all $x \ne 0$

So $-x^4 \le x^4 \sin\left(\frac{\pi}{x}\right) \le x^4$ for all $x \ne 0$ (b/c $x^4 > 0$)

now $\lim_{x\to 0} -x^4 = 0$, $\lim_{x\to 0} x^4 = 0$

So, $\lim_{x\to 0} x^4 \sin\left(\frac{\pi}{x}\right) = 0$ by the squeeze theorem.

(cf. the class notes)

(g)

$$\lim_{t \to 1} f(t),$$

where

$$f(t) = \begin{cases} -t+2, & t<1 \\ 2, & t=1 \\ t^2 + \log(t), & t>1 \end{cases}$$

consider 2 sided limits.

$$\lim_{t \to 1^-} f(t) = \lim_{t \to 1^-} -t+2 = -1+2 = 1$$

$$\lim_{t \to 1^+} f(t) = \lim_{t \to 1^+} t^2 + \log t = 1^2 + \log(1) = 1+0 = 1$$

So $\displaystyle\lim_{t \to 1} f(t) = 1.$

3. (5 points) This question is about continuity.

(a) (1 points) State the definition of continuity of $y = f(x)$ at $x = 3$.

$$\lim_{x \to 3} f(x) = f(3).$$

(b) (4 points) At what points is the function

$$f(x) = \begin{cases} \frac{x}{x-1}, & x < 0 \\ e^{x^2} + 3, & x \geq 0 \end{cases}$$

continuous? State your answer as an interval.

observe $\frac{x}{x-1}$ is continuous except at $x=1$, since we only use that formula for $x<0$, f is cts for $x<0$.

$e^{x^2}+3$ is cts everywhere, so certainly for $x>0$.

The only issue is at $x=0$.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x}{x-1} = \frac{0}{0-1} = 0$$

not equal.

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{x^3}+3 = e^0+3 = 4$$

so f is not cts at $x=0$.

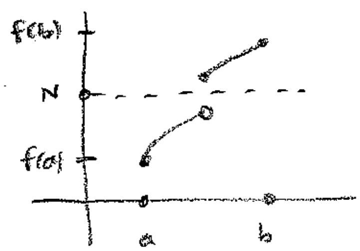
Conclude: $(-\infty, 0) \cup (0, \infty)$ are the pts. of cty.

4. (10 points) This question is about the Intermediate value theorem.

(a) (2 points) State the Intermediate value theorem precisely.

Let f be continuous on $[a,b]$. Suppose $f(a) \neq f(b)$. For every N between $f(a)$ and $f(b)$ there is a number c with $a<c<b$ and $f(c)=N$.

(b) (3 points) Sketch the graph of a function that shows the statement of the theorem is false if you omit the hypothesis that the function is continuous. Explain your graph.



Here f is a function on $[a,b]$ that is not cts.
Notice for the N shown that there is no c in (a,b) where $f(c)=N$.

(c) (5 points) Prove that there is a real-valued solution to the equation

$$\sqrt[4]{x} = 1 - x.$$

Set $f(x) = \sqrt[4]{x} - 1 + x$. We're done if we can find a root of f.

Notice f is continuous for $[0,\infty)$.

Notice also $f(0) = \sqrt[4]{0} - 1 + 0 = -1 < 0$

$f(1) = \sqrt[4]{1} - 1 + 1 = 1 > 0$

Since f is continuous on $[0,1]$ and changes sign at the endpoints, there must be a c with $0<c<1$ and $f(c)=0$ by the I.V.T.

5. (10 points) Consider the parametric equations given by

$$\begin{cases} x(t) &= \sin(t) \\ y(t) &= \cos^2(t), \end{cases}$$

for $0 \leq t \leq \pi$.

(a) (5 points) Find the Cartesian equations of the curve.

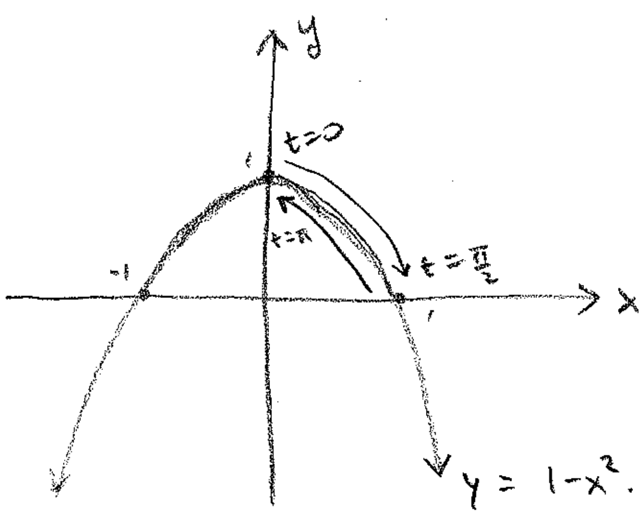
eliminate t.

$$\left.\begin{array}{l} x^2 = \sin^2(t) \\ y = \cos^2(t) \end{array}\right\} \text{add} \quad x^2 + y = 1$$

$y = 1 - x^2$ (a parabola!)

when $0 \leq t \leq \pi$, $0 \leq x \leq 1$.

(b) (5 points) Sketch as accurately as possible the curve traced out by the parametric equations. Label your axes carefully.



$t=0$	$(0,1)$
$t=\frac{\pi}{2}$	$(1,0)$
$t=\pi$	$(0,1)$

6. (10 points) Determine if the following statements are true or false. No justification is needed.

(a) (2 points) The domain of $\log_7(1-x^2)$ is $x<1$.

False. (notice $x=-2$ does not work)

(b) (2 points) $g(x)=\sqrt{x^2+9}$ is a polynomial.

False. $\left(\sqrt{x^2+9} \neq \sqrt{x^2}+\sqrt{9} = x+3\right)$

bad algebra

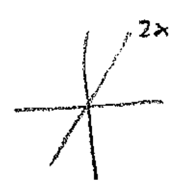
(c) (2 points) $f(x)=\frac{1}{x^2+2x+\sqrt{2}}$ is a rational function.

True. ($\sqrt{2}$ is just a coefficient)

(d) (2 points) The following equation holds for all real x:

$$\frac{x^2+2x+1}{x+1}=x+1.$$

False. (does it make sense to plug in $x=-1$?)



(e) (2 points) If f is one-to-one then $f^{-1}(x)=\frac{1}{f(x)}$.

False. $f(x)=2x$ is one-to-one (passes horiz. line test)

but $f^{-1}(x)=\frac{1}{2}\cdot x=\frac{x}{2}$

and $\frac{1}{f(x)}=\frac{1}{2x}$