Math 19: Fall 2013
Final Exam

NAME: SOLUTIONS

LECTURE:

Time: **3 hours**

This is a closed book and closed notes exam. Calculators and any electronic aid are not allowed.

For each problem, you should write down all of your work carefully and legibly to receive full credit. When asked to justify your answer, you should use theorems and/or mathematical reasoning to support your answer, as appropriate.

I understand and accept the provisions of the Stanford Honor Code.

Signature: ______________________________

Problem	Value	Score
1	6	
2	12	
3	8	
4	5	
5	5	
6	5	
7	5	
8	10	
9	6	
10	6	
11	4	
12	11	
13	17	
TOTAL	100	

Limit laws

Throughout, let a and c be real numbers, and suppose that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.

1. $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided that $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a}(f(x))^n = (\lim_{x\to a} f(x))^n$, for n a positive integer
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$, for n a positive integer
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, for n a positive integer, and provided that $a \geq 0$ if n is even
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, for n a positive integer, and provided that $\lim_{x\to a} f(x) \geq 0$ if n is even

Some derivatives

- $\dfrac{d}{dx}\arcsin x = \dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx}\arccos x = -\dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx}\arctan x = \dfrac{1}{1-x^2}$

Problem 1 : (6 points) Evaluate the following limits, if they exist. At every step, justify your work with a limit rule or a theorem.

a) $\lim\limits_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2}$

plug in: $\frac{\frac{1}{2} - \frac{1}{2}}{2-2} = \frac{0}{0}$

$$= \lim_{x \to 2} \frac{\frac{2}{2x} - \frac{x}{2x}}{x-2}$$

$$= \lim_{x \to 2} \frac{2-x}{2x} \div \frac{x-2}{1}$$

using rules 3,5,7,8

$$= \lim_{x \to 2} \frac{2-x}{2x} \cdot \frac{1}{x-2} = \lim_{x \to 2} -\frac{1}{2x} = -\frac{1}{4}$$

b) $\lim\limits_{x \to \infty} \frac{2x^2 - 3x}{3x^2 + 500}$

plug in: $\frac{\infty}{\infty}$

$$= \lim_{x \to \infty} \frac{x^2 \left(2 - \frac{3}{x}\right)}{x^2 \left(3 + \frac{500}{x^2}\right)}$$

$$= \lim_{x \to \infty} \frac{2 - \frac{3}{x}}{3 + \frac{500}{x^2}} = \frac{2}{3}$$

using rules 1,2,5,7

Problem 2 : (12 points) For each of the following expressions, compute $\frac{dy}{dx}$. You do not need to simplify your answer, but you do need to solve for $\frac{dy}{dx}$.

a) **[Section 3.2 # 27]** $y = \frac{x^2}{1+2x}$

$$\frac{dy}{dx} = \frac{2x(1+2x) - x^2(2)}{(1+2x)^2}$$

b) **[Section 3.5 # 13]** $e^{x/y} = x - y$

$$e^{x/y}\left(\frac{1 \cdot y - x \cdot \frac{dy}{dx}}{y^2}\right) = 1 - \frac{dy}{dx}$$

$$\frac{e^{x/y}}{y} - \frac{e^{x/y}x}{y^2}\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx}\left(1 - \frac{e^{x/y}x}{y^2}\right) = 1 - \frac{e^{x/y}}{y}$$

$$\frac{dy}{dx} = \frac{1 - \frac{e^{x/y}}{y}}{1 - \frac{e^{x/y}x}{y^2}}$$

c) **[Section 3.6 # 21]** $y = \sqrt{1-x^2}\arccos(x)$

$$\frac{dy}{dx} = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)\arccos x$$

$$+ \sqrt{1-x^2} \cdot \frac{-1}{\sqrt{1-x^2}}$$

d) **[Section 3.7 # 39]** $y = (\cos x)^x$

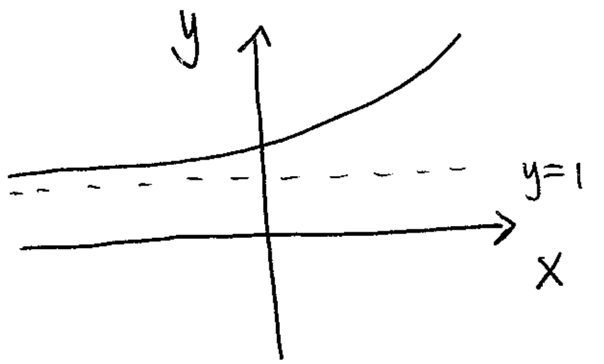
$$\ln y = \ln((\cos x)^x) = x \ln(\cos x)$$

$$\frac{1}{y}\frac{dy}{dx} = 1 \cdot \ln(\cos x) + x \frac{1}{\cos x}(-\sin x)$$

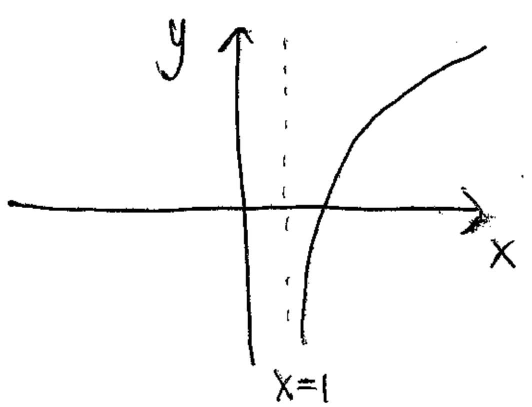
$$\frac{dy}{dx} = (\cos x)^x \left[\ln(\cos x) - \frac{x \sin x}{\cos x}\right]$$

Problem 3 : (8 points) Sketch a picture of the graph and use it to evaluate the following limits. NOTE: You must sketch the graph to receive credit.

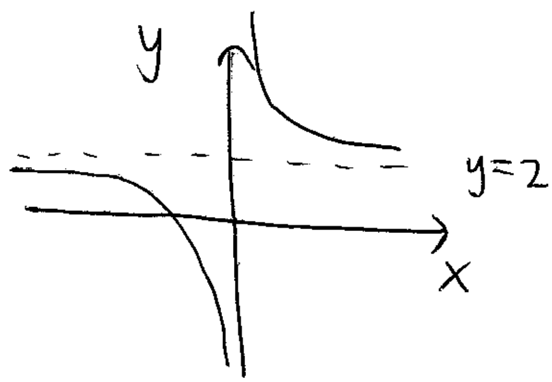
a) $\lim_{x \to -\infty} (e^x + 1) = 1$



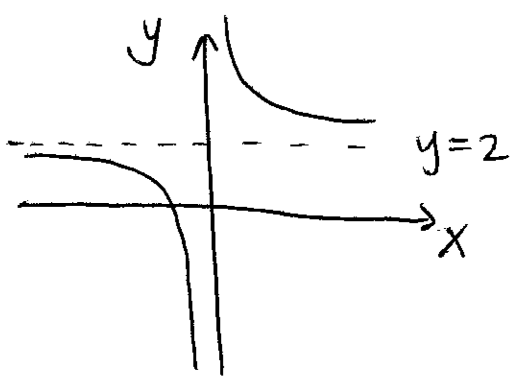
b) $\lim_{x \to 1^+} \ln(x - 1) = -\infty$



c) $\lim_{x \to 0^-} \left(\frac{1}{x} + 2 \right) = -\infty$



d) $\lim_{x \to \infty} \left(\frac{1}{x} + 2 \right) = 2$



Problem 4 : (5 points)

a) (2 points) Write down the definition for the following statement: The function f is differentiable at $x = a$.

The limit $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ exists.

b) (3 points) Use your definition above to determine if $f \colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is differentiable at $x = 2$.

$$\lim_{h \to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(2+h)^2 - 4}{h}$$

$$= \lim_{h \to 0} \frac{4+4h+h^2-4}{h}$$

$$= \lim_{h \to 0} \frac{h(4+h)}{h}$$

$$= \lim_{h \to 0} (4+h) = 4$$

f is differentiable at $x=2$

Problem 5 : (5 points) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by the rule

$$f(x) = \begin{cases} x^2 + 2x + 3 & \text{if } x \le 1, \\ 5x + 1 & \text{if } x > 1. \end{cases}$$

a) (3 points) Is f continuous at $x = 1$? Justify using the definition of continuity.

$f(1) = 1+2+3 = 6$

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} (5x+1) = 6$

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} (x^2+2x+3) = 6$

$f(1) = \lim_{x\to1} f(x)$, f is continuous at $x=1$

b) (2 points) Is f differentiable at $x = 1$? Justify using the definition of differentiability.

$$\lim_{h\to0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to0^+} \frac{5(1+h)+1-6}{h}$$

$$= \lim_{h\to0^+} \frac{5+5h+1-6}{h} = \lim_{h\to0^+} 5 = 5$$

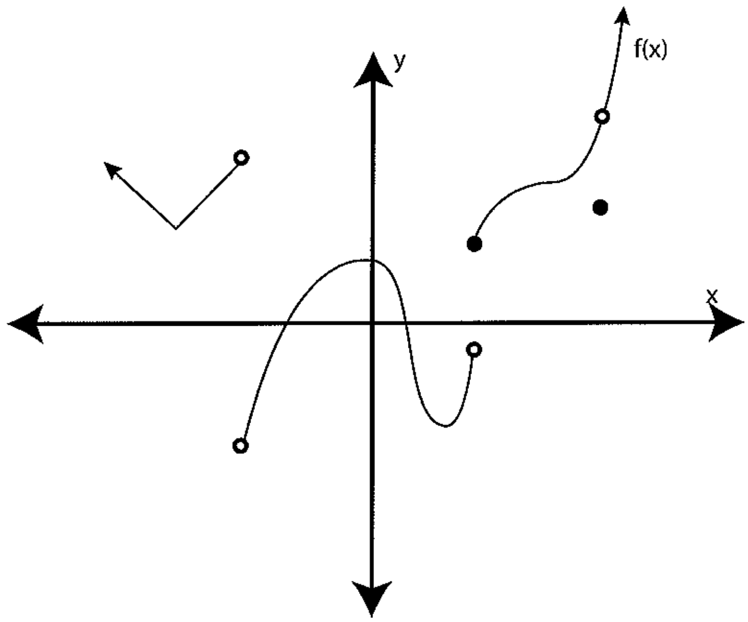
$$\lim_{h\to0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to0^-} \frac{(1+h)^2+2(1+h)+3-6}{h}$$

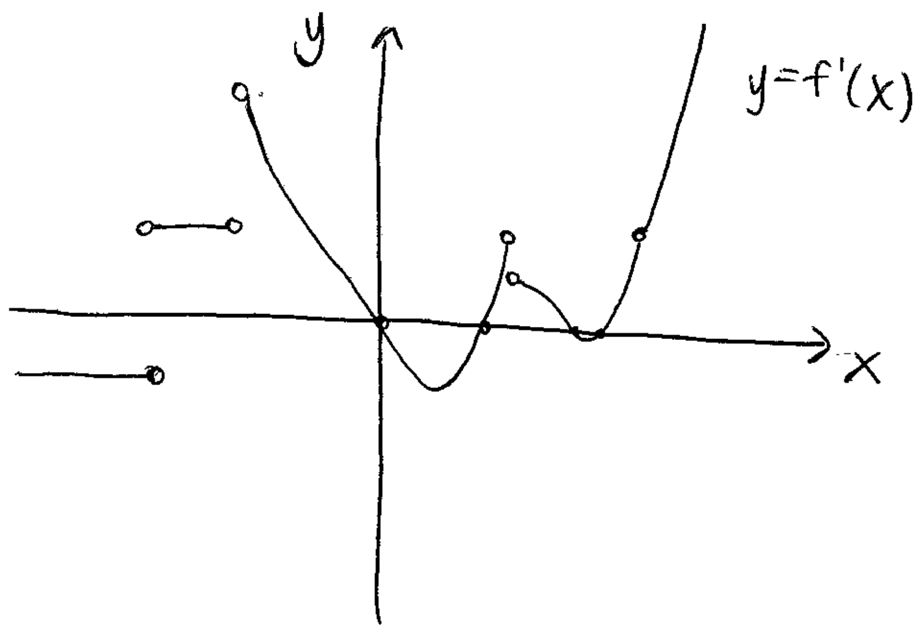
$$= \lim_{h\to0^-} \frac{1+2h+h^2+2+2h+3-6}{h}$$

$$= \lim_{h\to0^-} \frac{h(2+h+2)}{h} = 4$$

The limit does not exist; f is not differentiable at $x=1$.

Problem 6 : (5 points) Given this graph of f sketch a plausible graph for f'





Problem 7 : (5 points) Suppose that a certain commodity has revenue function

$$R\colon [0,4] \to \mathbb{R}, \quad R(x) = -x^3 + 3x^2 + 9x.$$

What is the maximum revenue?

critical points; $R'(x) = -3x^2 + 3 \cdot 2x + 9$

$$= -3x^2 + 6x + 9$$
$$= -3(x^2 - 2x - 3)$$
$$= -3(x-3)(x+1)$$

$$x = 3, \quad x = -1$$

$x=-1$ is not in the domain so we reject it.

Since f is continuous on a closed interval, its global maximum is among the end points and critical points.

$f(0) = 0$

$f(3) = -27 + 27 + 27 = 27$

$f(4) = -64 + 48 + 36 = 20$

The maximum revenue is 27.

Problem 8 : (10 points) Let $f\colon [0,\infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$. We have that

$$f(3) = \sqrt{3} \approx 1.73205080757.$$

a) (3 points) Compute the equation of the line tangent to f at $x = 4$.

$f(4) = \sqrt{4} = 2$

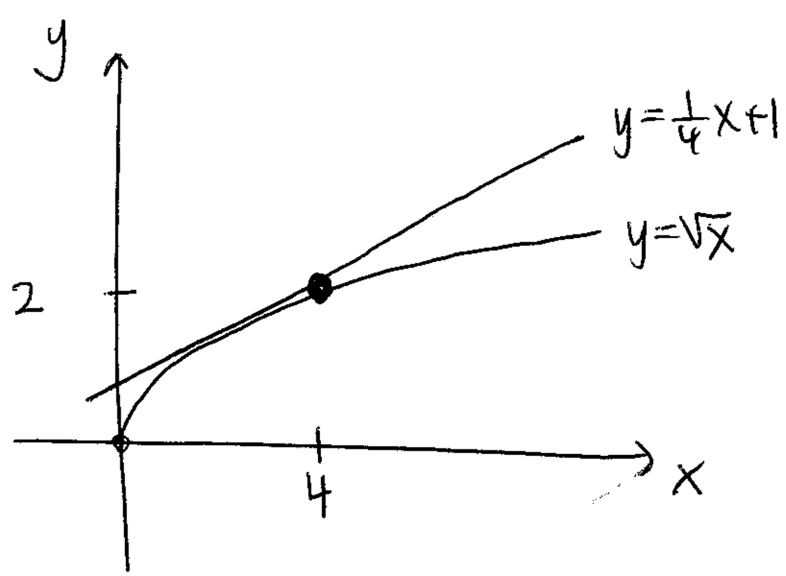
$f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$

$f'(4) = \frac{1}{2} 4^{-\frac{1}{2}} = \frac{1}{4}$

The tangent line has slope $\frac{1}{4}$ and goes through $(4,2)$:

$2 = \frac{1}{4} 4 + b$ $\qquad$ $b = 1$ $\qquad$ $y = \frac{1}{4}x + 1$

b) (2 points) Graph f along with its tangent line at $x = 4$.



c) (2 points) The tangent line you found in a) is the graph of a function L. Give the **function** L.

$L: \mathbb{R} \to \mathbb{R}$

$L(x) = \frac{1}{4}x + 1$

d) (3 points) When x is near 4, $L(x)$ is near $f(x)$. In particular,

$$L(3) \approx f(3) = \sqrt{3}.$$

Compute $L(3)$, and see how well it agrees with the value of $\sqrt{3}$ above.

$L(3) = \frac{3}{4} + 1 = 1.75 \approx 1.73\ldots$

not bad!

Recall that

$$\frac{d}{dx}2^x = \ln 2 \cdot 2^x.$$

Problem 9 : (6 points) Let $y = \log_2(x)$. Compute $\frac{dy}{dx}$ using implicit differentiation.

$$y = \log_2(x)$$

$$x = 2^y$$

$$1 = \ln 2 \cdot 2^y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\ln 2 \cdot 2^y} = \frac{1}{\ln 2 \cdot x}$$

Problem 10 : (6 points)
One day Dr. Campisi and Dr. Vincent went out for a drive. The FasTrak in their car registered them crossing the Richmond-San Rafael bridge at 12:05pm and it registered them crossing the Golden Gate bridge, 20 miles away, at 12:20pm. Assume that the position function of the car is differentiable at each time during the drive. Two weeks later they received a ticket in the mail, citing them for exceeding the speed limit of 60 miles per hour.

a) (2 points) State the Mean Value Theorem.

Let f be a differentiable function on $[a,b]$.
Then there is c in $[a,b]$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

b) (4 points) Show that Dr. Campisi and Dr. Vincent did in fact exceed the speed limit at some point during their drive. Be sure to clearly define any variable or functions that you use.

Let s be their position function

$$s(12{:}05)=0$$
$$s(12{:}20)=20$$

there is a time c between 12:05 and 12:20 such that

$$s'(c) = \frac{s(12{:}20)-s(12{:}05)}{1/4} = 20\cdot 4 = 80 \text{ mph}$$

15 mins in hours!

They went really fast.

Problem 11 : (4 points) Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be two differentiable functions such that

$$g(0) = \pi, \quad g'(0) = 3, \quad f(0) = 0, \quad \text{and} \quad f'(0) = 0.$$

Let B be a function with rule

$$B(x) = \cos(g(\sin(f(x)))).$$

Compute $B'(0)$.

$$B'(x) = -\sin(g(\sin(f(x)))) \cdot g'(\sin(f(x))) \cdot \cos(f(x)) \cdot f'(x)$$

$$B'(0) = -\sin(g(\sin(f(0)))) \cdot g'(\sin(f(0))) \cdot \cos(f(0)) \cdot f'(0)$$

$= 0$ since $f'(0) = 0$.

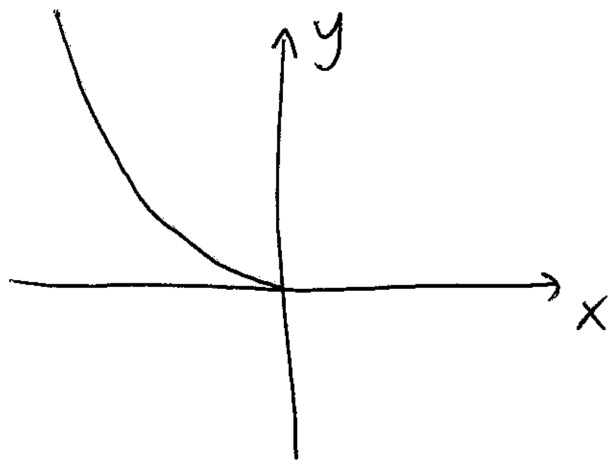
Problem 12 : (11 points) Consider the function

$$f\colon (-\infty, 0] \to \mathbb{R}, \qquad f(x) = x^2.$$

a) (1 point) What is the domain of f?

$(-\infty, 0]$

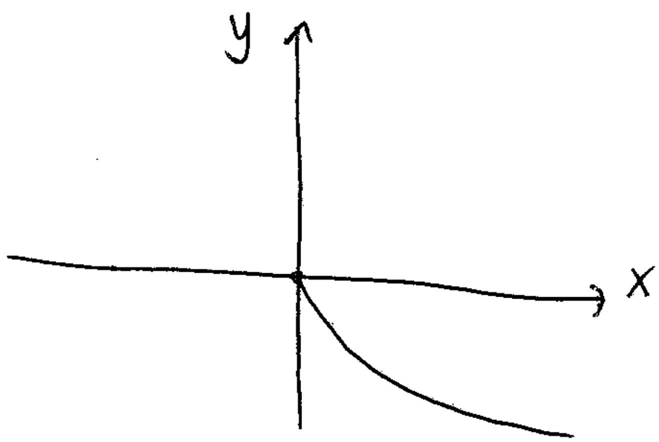
b) (2 points) Sketch the graph of f.



c) (1 point) Is f one-to-one? Please simply answer yes or no.

yes

d) (2 points) Sketch the graph of f^{-1}.



e) (2 points) Give the rule of the function f^{-1}. Hint: Its graph is a transformation of the graph of a function you know.

$$f^{-1}(x) = -\sqrt{x}$$

f) (1 point) Let $x < 0$. Compute $f^{-1}(f(x))$.

$$f^{-1}(f(x)) = f^{-1}(x^2) = -\sqrt{x^2} = -|x| = x$$

g) (1 point) Let $x > 0$. Compute $f^{-1}(f(x))$.

$x > 0$ is not in the domain of f!

also accepted:

$$f^{-1}(f(x)) = f^{-1}(x^2) = -\sqrt{x^2} = -|x| = -x$$

h) (1 point) Compute $f(f^{-1}(x))$.

$$f(-\sqrt{x}) = (-\sqrt{x})^2 = x$$

Problem 13 : (17 points) Let $f(x) = x^3 - 3x^2$.

a) (1 point) State the domain of f.

$\mathbb{R}$

b) (2 points) Determine all x- and y-intercepts.

$f(0) = 0$ y-intercept is $(0,0)$

$0 = x^3 - 3x^2 = x^2(x-3)$ x-intercepts are

$(3,0)$
$(0,0)$

c) (2 points) Find the equations for all vertical and horizontal asymptotes. If there are none, please write "none."

there are no vertical asymptotes since f is a polynomial function.

$$\lim_{x\to\infty} (x^3 - 3x^2) = \lim_{x\to\infty} x^3\left(1 - \frac{3}{x}\right) = \infty$$

$$\lim_{x\to-\infty} (x^3 - 3x^2) = \lim_{x\to-\infty} x^3\left(1 - \frac{3}{x}\right) = -\infty$$

no horizontal asymptotes

d) (3 points) Find all values for which f is increasing and decreasing.

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

x	-1	0	1	2	3
$f'(x)$	$+$		$-$		$+$
$f(x)$	$\nearrow$		$\searrow$		$\nearrow$

increasing: $(-\infty, 0) \cup (2, \infty)$
decreasing: $(0, 2)$

e) (1 point) Determine all maximum and minimum values of f.

local/relative max at $(0,0)$
local/relative min at $(2,-4)$

f) (3 points) Find all values for which f is concave up and concave down.

$$f''(x) = 6x - 6 = 6(x-1)$$

x	0	1	2
$f''(x)$	$-$		$+$
$f(x)$	$\frown$		$\smile$

concave down: $(-\infty, 1)$
concave up: $(1, \infty)$

g) (1 point) Determine all inflections points.

$(1, -2)$

h) (4 points) Neatly sketch the graph of f. Label all asymptotes with their equation, and label all special points (intercepts, holes, maxima, minima, points of inflection) with their coordinates.