Math 19: Fall 2012
Final Exam

NAME: SOLUTIONS.

LECTURE:

Time: **3 hours**

This is a closed book and closed notes exam. Calculators and any electronic aid are not allowed.

For each problem, you should write down all of your work carefully and legibly to receive full credit. When asked to justify your answer, you should use theorems and/or mathematical reasoning to support your answer, as appropriate.

I understand and accept the provisions of the Stanford Honor Code.

Signature: ______________________________

Problem	Value	Score
1	10	
2	8	
3	12	
4	12	
5	13	
6	10	
7	10	
8	10	
9	15	
TOTAL	100	

Some derivatives

- $\dfrac{d}{dx} \arcsin x = \dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx} \arccos x = -\dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx} \arctan x = \dfrac{1}{1-x^2}$

Problem 1 : (10 points)

a) (4 points) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function that is differentiable everywhere. Write the definition of the linearization of f at the point a.

$L: \mathbb{R} \to \mathbb{R}$

$L(x) = f(a) + f'(a)(x-a)$

b) (3 points) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is given by the rule $f(x) = 2x + 3$. Find the linearization of f at $a = 4$.

$f(a) = f(4) = 11$

$f'(x) = 2$

$f'(a) = f'(4) = 2$

$L: \mathbb{R} \to \mathbb{R}$

$L(x) = 11 + 2(x-4)$

c) (3 points) Now suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is given by the rule $f(x) = mx + b$, for some real numbers m and b. Find the linearization of f at a. Simplify your answer.

$f(a) = ma + b$

$f'(x) = m$

$f'(a) = m$

$$
\begin{aligned}
L(x) &= ma + b + m(x-a) \\
&= ma + b + mx - ma \\
&= mx + b
\end{aligned}
$$

$L: \mathbb{R} \to \mathbb{R}$

$L(x) = mx + b$

$\Rightarrow$ A line is its own linearization!

Problem 2 : (8 points) Compute or simplify the following quantities, as appropriate.

a) (2 points) $\dfrac{\sqrt[3]{\sqrt{b}\,a^2}}{b\sqrt{a}} = \dfrac{(b^{1/2}a^2)^{1/3}}{b a^{1/2}} = \dfrac{b^{1/6}a^{2/3}}{b a^{1/2}} = \dfrac{a^{1/6}}{b^{5/6}}$

$\frac{2}{3} - \frac{1}{2} = \frac{4}{6} - \frac{3}{6} = \frac{1}{6}$

b) (2 points) $\dfrac{x^{-s+1}(x^2)^{s+t}}{x^{-4t-2}} = \dfrac{x^{-s+1}x^{2s+2t}}{x^{-4t-2}} = x^{s+6t+3}$

$(-s+1)+(2s+2t)-(-4t-2)$

$= -s+1+2s+2t+4t+2 = s+6t+3$

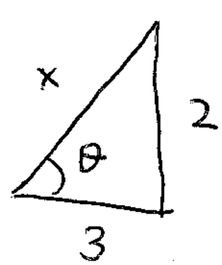
c) (2 points) $e^{5\ln x + \ln(x-1) - \ln(x-2)}$

$= e^{\ln\left(\frac{x^5(x-1)}{(x-2)}\right)} = \dfrac{x^5(x-1)}{x-2}$

d) (2 points) $\sin\left(\arctan\frac{2}{3}\right) = \sin\theta = \dfrac{2}{\sqrt{13}}$

$\theta = \arctan\frac{2}{3}$

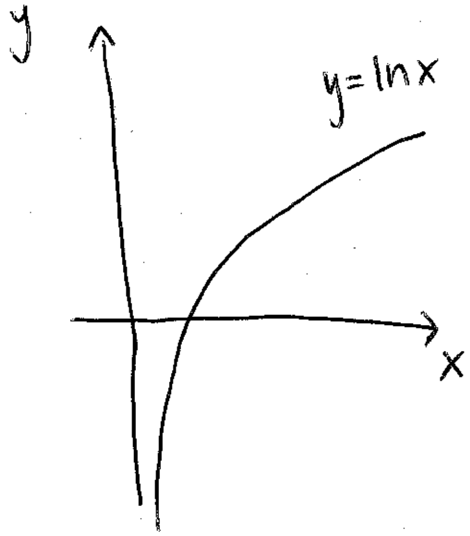
$\tan\theta = \frac{2}{3}$



$x = \sqrt{3^2+2^2} = \sqrt{9+4} = \sqrt{13}$

Problem 3 : (12 points) Compute the following limits:

a) (3 points) For this limit only, graph the basic function and use it to determine what the limit is: $\lim_{x \to \infty} \ln x$



$$\lim_{x \to \infty} \ln x = \infty$$

b) (3 points) $\lim_{x \to 2^-} \frac{x^2 - 2x - 3}{x^2 + x - 6}$

$\frac{-3}{0}$ → this will be $\pm\infty$

$$= \lim_{x \to 2^-} \frac{x^2 - 2x - 3}{(x+3)(x-2)} = \frac{-3}{5 \cdot 0^-} = +\infty$$

c) (3 points) $\lim_{\theta \to 0} \frac{\sin 6\theta}{\theta}$ $= \lim_{\theta \to 0} \frac{6 \sin 6\theta}{6\theta}$

$= 6 \lim_{\theta \to 0} \frac{\sin 6\theta}{6\theta} = 6$

d) (3 points) Let $f \colon [0, 2] \to \mathbb{R}$. Assume that $3x \leq f(x) \leq x^3 + 2$ for all x in the domain of f. Calculate $\lim_{x \to 1} f(x)$.

$\lim_{x \to 1} 3x = 3$

$\lim_{x \to 1} (x^3+2) = 3$

By the Squeeze Theorem

$\lim_{x \to 1} f(x) = 3$

Problem 4 : (12 points) In each of the following problems, let y be a function of x. Compute the rule for $\frac{dy}{dx}$.

a) (3 points) $y\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$, $y = \dfrac{e^x}{x}$

$$\frac{dy}{dx} = \frac{e^x \cdot x - 1 \cdot e^x}{x^2}$$

b) (3 points) $y\colon [-1,1] \to \mathbb{R}$, $y = (\arccos x)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(\arccos x)^{-\frac{1}{2}} \cdot \frac{-1}{\sqrt{1-x^2}}$$

c) (3 points) $y\colon \mathbb{R} \to \mathbb{R}$, $xy^2 + e^y = x^3$

$$1 \cdot y^2 + x \cdot 2y \frac{dy}{dx} + e^y \frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx}\left(2xy + e^y\right) = 3x^2 - y^2$$

$$\frac{dy}{dx} = \frac{3x^2 - y^2}{2xy + e^y}$$

d) (3 points) $y\colon (5/3, \infty) \to \mathbb{R}$, $y = \dfrac{(5x-3)^{11}(2x^3+1)^{1/2}}{(-3x+5)^3}$

$$\ln y = 11 \ln(5x-3) + \frac{1}{2}\ln(2x^3+1) - 3\ln(-3x+5)$$

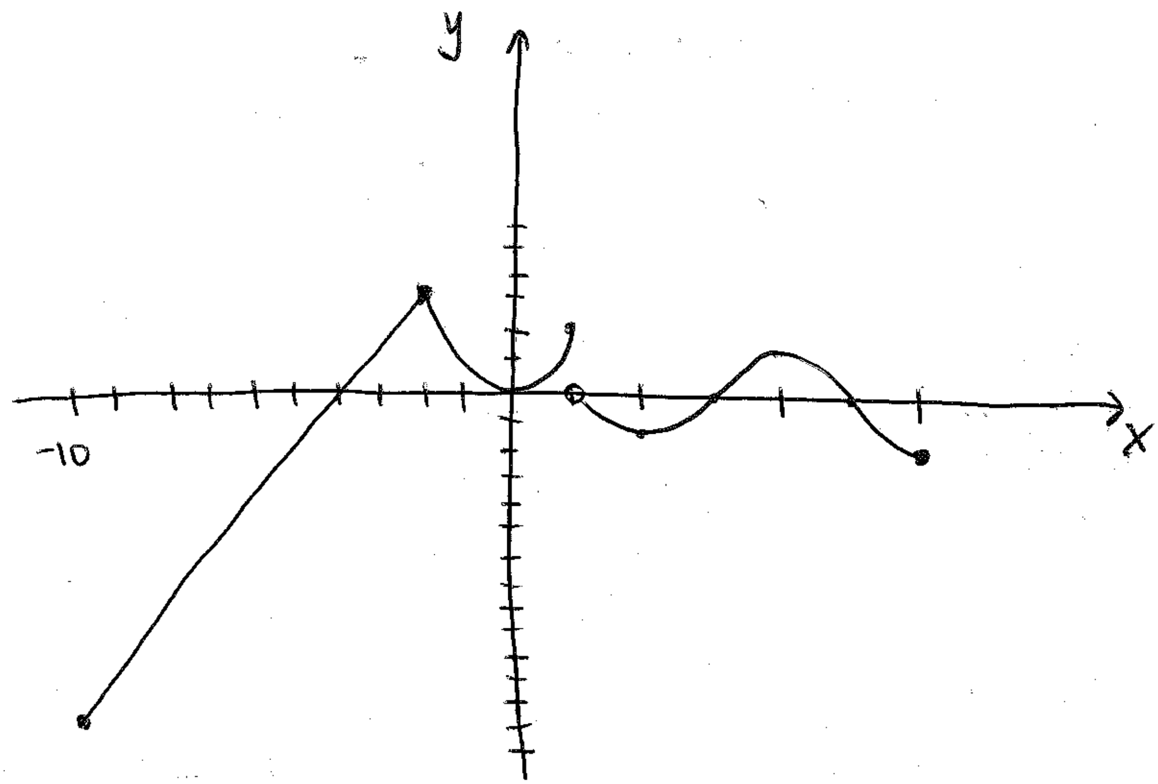
$$\frac{1}{y}\frac{dy}{dx} = \frac{11}{5x-3} \cdot 5 + \frac{1}{2} \frac{1}{2x^3+1} \cdot 6x^2 - \frac{3}{-3x+5}(-3)$$

$$\frac{dy}{dx} = \frac{(5x-3)^{11}(2x^3+1)^{1/2}}{(-3x+5)^3}\left(\frac{55}{5x-3} + \frac{3x^2}{2x^3+1} + \frac{9}{-3x+5}\right)$$

Problem 5 : (13 points) Consider the function $f\colon [-10, 3\pi] \to \mathbb{R}$ given by the rule

$$f(x) = \begin{cases} 2x+8 & \text{if } -10 \le x \le -2, \\ x^2 & \text{if } -2 < x \le \pi/2, \\ \cos x & \text{if } \pi/2 < x \le 3\pi. \end{cases}$$

a) (3 points) Sketch the graph of f.



b) (2 points) Is f continuous at $x = \pi/2$? Justify.

No: $\lim\limits_{x \to \frac{\pi}{2}^-} f(x) = \lim\limits_{x \to \frac{\pi}{2}^-} x^2 = \frac{\pi^2}{4}$

$\lim\limits_{x \to \frac{\pi}{2}^+} f(x) = \lim\limits_{x \to \frac{\pi}{2}^+} \cos x = 0$

$\lim\limits_{x \to \frac{\pi}{2}} f(x)$ does not exist.

Recall that f is given by the rule

$$f(x) = \begin{cases} 2x+8 & \text{if } -10 \le x \le -2, \\ x^2 & \text{if } -2 < x \le \pi/2, \\ \cos x & \text{if } \pi/2 < x \le 3\pi. \end{cases}$$

c) (4 points) Is f continuous at $x = -2$? Justify.

Yes: $f(-2) = 2(-2)+8 = 4$

$$\left.\begin{aligned} \lim_{x \to -2^-} f(x) &= \lim_{x \to -2^-} (2x+8) = 4 \\ \lim_{x \to -2^+} f(x) &= \lim_{x \to -2^+} x^2 = 4 \end{aligned}\right\} \lim_{x \to 2} f(x) = 4$$

and it exists

Also $f(2) = \lim_{x \to 2} f(x)$.

d) (4 points) Is f differentiable at $x = -2$? Justify.

No: $\displaystyle \lim_{h \to 0^-} \frac{f(-2+h) - f(-2)}{h}$

$$= \lim_{h \to 0^-} \frac{2(-2+h)+8-4}{h} = \lim_{h \to 0^-} \frac{2h}{h} = 2$$

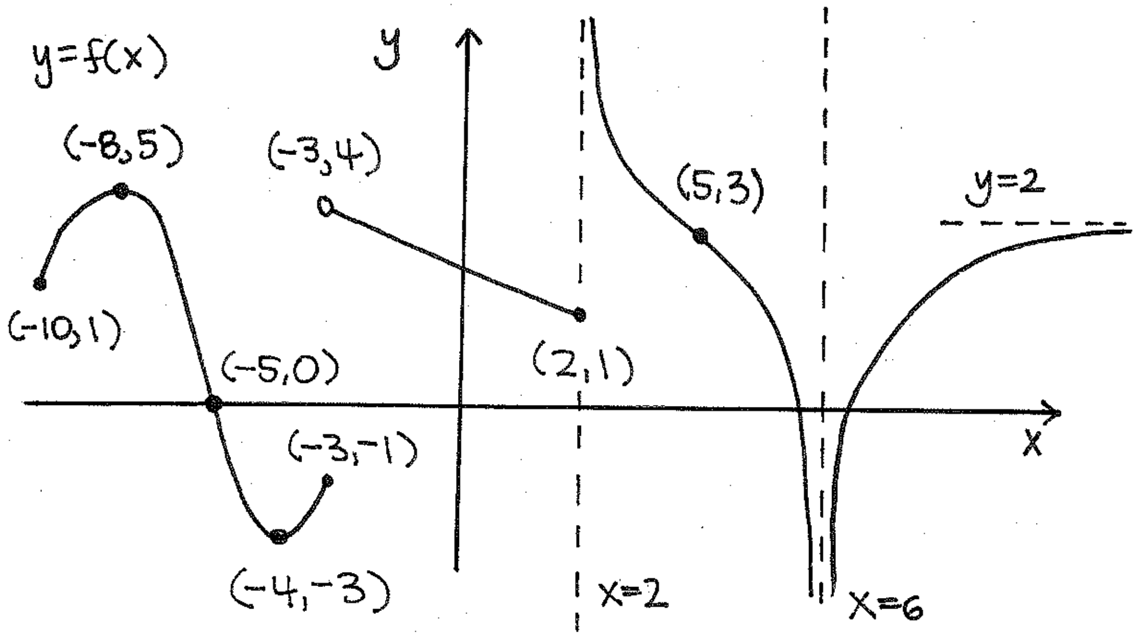
$$\lim_{h \to 0^+} \frac{f(-2+h) - f(-2)}{h}$$

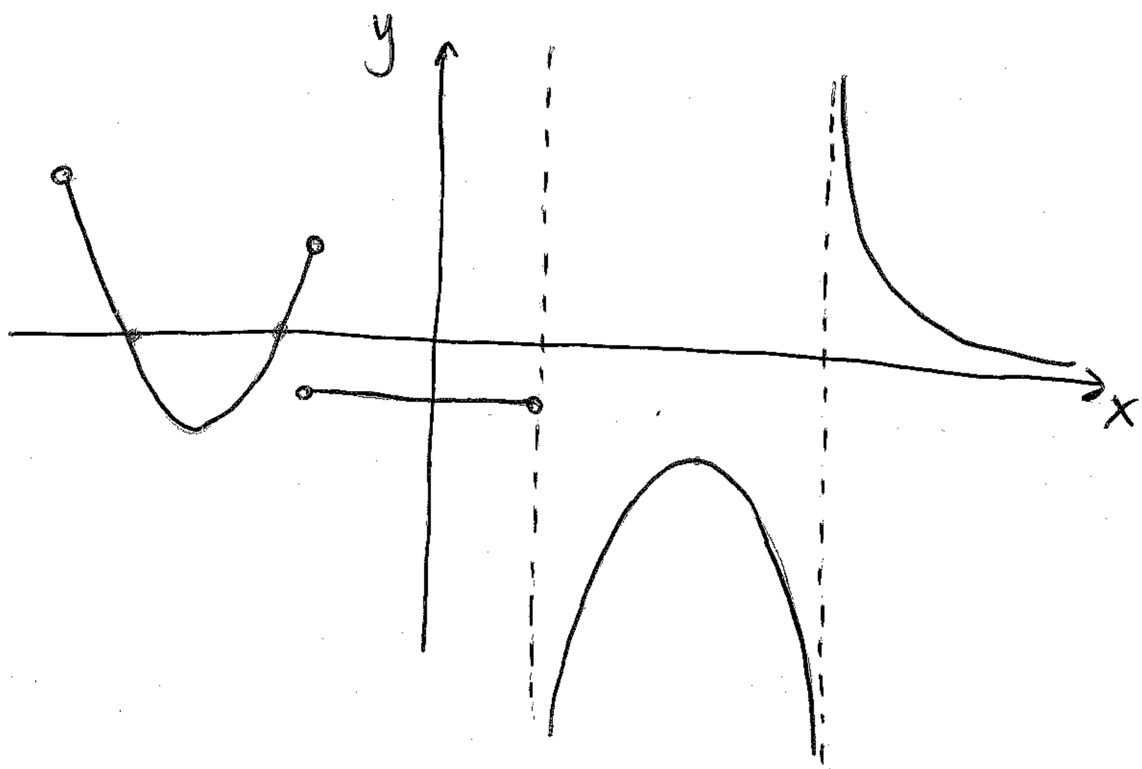
$$= \lim_{h \to 0^+} \frac{(-2+h)^2 - 4}{h} = \lim_{h \to 0^+} \frac{4 - 4h + h^2 - 4}{h}$$

$$= \lim_{h \to 0^+} (h-4) = -4$$

$\displaystyle \lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h}$ does not exist so f is not differentiable at $x=-2$.

Problem 6 : (10 points) Consider the function f whose graph is given below. Sketch a plausible graph for the function f'. Be as accurate as you can.





Problem 7 : (10 points) The cost, in dollars, of producing x calculus textbooks is given by the cost function $C\colon [0,\infty) \to \mathbb{R}$ with rule

$$C(x) = x^3 - 6x^2 + 13x.$$

a) (2 points) Find the rule for marginal cost function.

$$C'(x) = 3x^2 - 12x + 13$$

b) (4 points) Find the x-coordinate of the inflection point of C.

$$C''(x) = 6x - 12$$

$C''(x) = 0$ when $x = 2$

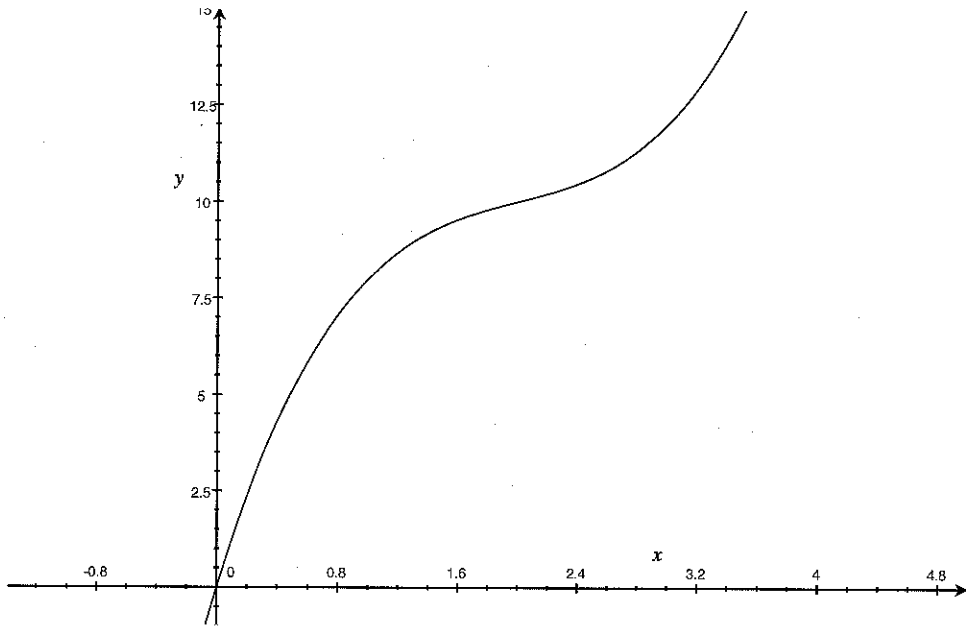
$C''(1) = -6 < 0$
$C''(3) = 6 > 0$

so $x = 2$ is an inflection point.

Recall that the cost function is given by the rule

$$C(x) = x^3 - 6x^2 + 13x.$$

This is the graph of C:



c) (2 points) What is the significance of the inflection point you found in b)?

When $x<2$, the marginal cost is decreasing (more and more economies of scale)

When $x>2$, the marginal cost is increasing.

Problem 8 : (10 points)

a) (5 points) Use implicit differentiation and the fact that $\frac{d}{dx}e^x = e^x$ to show that

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

$y = \ln x$

$x = e^y$

$1 = e^y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$

b) (5 points) Let D be a subset of the real numbers and suppose that $f\colon D \to \mathbb{R}$ is a one-to-one differentiable function such that its inverse f^{-1} is also differentiable. Use implicit differentiation to show that

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))}$$

for all x such that $f'(f^{-1}(x))$ is not zero.

$$y = f^{-1}(x)$$

$$x = f(y)$$

$$1 = f'(y)\,\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

Problem 9 : (15 points) In this problem, parts a) and c) go together, and you can do part b) independently.

a) (3 points) State the Intermediate Value Theorem.

Let f be continuous on [a,b]. For any N between f(a) and f(b), there is c in [a,b] such that f(c) = N.

b) Consider the function $f\colon [0,1] \to \mathbb{R}$,

$$f(x) = x(x-1). = x^2 - x$$

(a) (3 points) f has an extremum (either a maximum or a minimum) between 0 and 1. Find its location (both x and y coordinates).

$f'(x) = 2x - 1$ $\quad f'(x) = 0$ when $x = \frac{1}{2}$

$f''(x) = 2$ $\quad$ so $x = \frac{1}{2}$ is a minimum.

$f(\frac{1}{2}) = (\frac{1}{2})^2 - \frac{1}{2} = \frac{1}{4} - \frac{2}{4} = -\frac{1}{4}$

(b) (2 points) Is the extremum a maximum or a minimum?

a minimum since $f''(\frac{1}{2}) > 0$

 ← minimum

c) Now consider the function $g\colon [0,1] \to \mathbb{R}$,

$$g(x) = x(x-1)(x-2)(x-3).$$

$$= (x^2-x)(x^2-5x+6) = x^4-5x^3+6x^2-x^3+5x^2-6x$$

(a) (3 points) Show that g has an extremum between 0 and 1. You do not have to give its location.

$g(x) = x^4 - 6x^3 + 11x^2 - 6x$

$g'(x) = 4x^3 - 18x^2 + 22x - 6$

$g'(0) = -6 < 0$

$g'(1) = 4 - 18 + 22 - 6$
$= 2 > 0$

g' is continuous on $[0,1]$

By the Intermediate Value Theorem, there is c in $[0,1]$ s.t. $g'(c) = 0$.

(b) (2 points) Is the extremum a maximum or a minimum?

g' goes from negative (g is decreasing)
to positive (g is increasing)

So the extremum is a minimum

(c) (2 points) Is the extremum to the left or to the right of $x = 1/2$?

$g'(\tfrac{1}{2}) = 4\left(\tfrac{1}{2}\right)^3 - 18\left(\tfrac{1}{2}\right)^2 + 22\left(\tfrac{1}{2}\right) - 6$

$= \frac{4}{8} - \frac{18}{4} + 11 - 6 = \frac{1}{2} - \frac{9}{2} + 5 = -4 + 5 = 1 > 0$

at $x = \frac{1}{2}$ g' is already positive. As before g' is continuous on $[0, \frac{1}{2}]$ and by the IVT there is c in $[0, \frac{1}{2}]$ such that $g'(c) = 0$. [In other words, to the left of $\frac{1}{2}$].