Math 108 Homework 6 Solutions

Problem 13G.
Show that (13.12) leads to $B(n) = \frac{1}{e}\sum_{k=0}^{\infty} \frac{k^n}{k!}$.

Answer: We actually just use (13.13) and (13.11). We get from (13.11) that $B(n) = \sum_{k=1}^{n} S(n,k)$, but note that for $k = 0$ and $k > n \geq 1$ that $S(n,k) = 0$, so we also have $B(n) = \sum_{k=0}^{\infty} S(n,k)$. Next, we plug in the formula from (13.13), and obtain

$$\begin{aligned}
B(n) &= \sum_{k=0}^{\infty} S(n,k) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n \\
&= \sum_{k=0}^{\infty} \sum_{i=0}^{k} (-1)^{k-i} \frac{1}{i!(k-i)!} i^n \\
&= \sum_{i=0}^{\infty} \sum_{k=i}^{\infty} (-1)^{k-i} \frac{1}{i!(k-i)!} i^n \\
&= \sum_{i=0}^{\infty} \frac{i^n}{i!} \sum_{k=i}^{\infty} \frac{(-1)^{k-i}}{(k-i)!} \\
&= \sum_{i=0}^{\infty} \frac{i^n}{i!} \sum_{k=0}^{\infty} \frac{(-1)^{k}}{k!} \\
&= \frac{1}{e} \sum_{i=0}^{\infty} \frac{i^n}{i!},
\end{aligned}$$

concluding the proof. □

Problem 14M.
Count pairs (f, g) of functions f, g, where f maps $\{1, 2, ..., r\}$ to $\{1, 2, ..., n\}$ and g is a permutation of $\{1, 2, ..., n\}$ that fixes the image of f pointwise. Doing this in two ways prove that for $n \geq r$

$$\sum_{k=1}^{n} \binom{n}{k} k^r d_{n-k} = B(r) \cdot n!,$$

where d_m denotes the number of derangements of $1, 2, ..., m$ and $B(r)$ is a Bell number.

Answer: First, consider the set of functions g with fixed point set $\{x \in \{1, ..., n\} \mid g(x) = x\}$ of size k. This collection of functions has size equal to $\binom{n}{k} d_{n-k}$, since there are $\binom{n}{k}$ ways to choose the fixed point set, and any permutation with precisely that fixed point set is a derangement when restricted to the remaining $n-k$ unfixed points. Next, the set of functions f which can be paired with a given function g which has precisely k fixed points is equal to the number of mappings of $\{1, ..., r\}$ into that set of k elements, and this is of size k^r. Thus, the number of such pairs (f, g) where g has k fixed points is equal to $\binom{n}{k} k^r d_{n-k}$. Summing this over all possible values of k, we get that the total number of pairs (f, g) of interest is equal to the left hand side of the desired identity,

$$\sum_{k=1}^{n} \binom{n}{k} k^r d_k.$$

We now count this set in another way. Consider the set of functions f with image of size i. For a given image set $\{x_1, ..., x_i\}$, there are $S(r, i)$ possible ways to split $\{1, ..., r\}$ into i subsets which will be the preimages of each of the values $x_1, ..., x_i$. For each of these $S(r, i)$ ways, there are $i!$ ways of assigning specific x_i values to the preimage sets. Next, there are $\binom{n}{i}$ possible choices of image set. Finally, given a specific image set of size i, the number of permutations g of $\{1, ..., n\}$ which fix at least that image set is just $(n-i)!$, since this is the number of permutations of the remaining elements of $\{1, ..., n\}$. Thus, the set of pairs (f, g) for which f has image of size i is equal to $S(r, i) i! \binom{n}{i} (n-i)! = S(r, i) n!$. Summing this over the possible values of i, we get that the set of pairs (f, g) has size

$$\sum_{i=1}^{r} S(r, i) n! = B(r) \cdot n!.$$

Thus, we have proved the desired identity. □

Problem 15A.
Show that $p_k(n) = p_{k1}(n1) + p_k(nk)$ and use this to prove (15.2).

Answer: The identity $p_k(n) = p_{k1}(n1) + p_k(nk)$ is obtained by splitting the set of partitions of n into k parts into two collections: those for which $x_k = 1$, and those for which $x_k > 1$. The set of partitions of n into k parts with $x_k = 1$ are in bijective correspondence with the set of partitions of $n-1$ into $k-1$ parts, and this correspondence is obtained simply by dropping the k^{th} part (i.e. the $x_k = 1$ part) from the original partition. Note that this quantity is equal to $p_{k-1}(n-1)$. Next, the set of partitions of n into k parts with $x_k > 1$ is in bijective correspondence with the set of partitions of $n-k$ into k parts by subtracting one from each of the parts (i.e., setting $y_k = x_k - 1$ gives the desired correspondence). This quantity is equal to $p_k(n-k)$. Thus we have shown that $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$.

Next, we prove (15.2):

$$p_k(n) = \sum_{s=1}^{k} p_s(n-k).$$

We prove this by induction, starting with the base case, $k = 1$. Note that $p_1(n) = 1$ for any choice of n, so for any n, we have $p_1(n) = p_1(n-1)$ and the base case is verified.

Next, suppose the claim holds for k. Then we have from the identity proved above that

$$p_{k+1}(n) = p_k(n-1) + p_{k+1}(n-(k+1)),$$

and applying the induction hypothesis, we get

$$\begin{aligned} p_{k+1}(n) &= p_k(n-1) + p_{k+1}(n-(k+1)) \\ &= \sum_{s=1}^{k} p_s(n-1-k) + p_{k+1}(n-(k+1)) \\ &= \sum_{s=1}^{k+1} p_s(n-(k+1)). \end{aligned}$$

Thus we have completed the induction step, and this induction argument verifies (15.2). □

Problem 15B.
We choose three vertices of a regular n-gon and consider the triangle they form. Prove directly that there are $\{\frac{1}{12}n^2\}$ mutually incongruent triangles that can be formed in this way, thereby providing a second proof of (15.5).

Answer: For this question, we will use the notation

$$1_{a|n} = \begin{cases} 1, & a \text{ divides } n \\ 0, & \text{else} \end{cases}.$$

Let $v_1, ..., v_n$ represent the n vertices in the regular n-gon in consecutive (e.g., clockwise) order starting with some vertex v_1. Note that two triangles obtained by taking three vertices $\{v_{i_1}, v_{i_2}, v_{i_3}\}$ and $\{v_{j_1}, v_{j_2}, v_{j_3}\}$ are congruent if and only if the sets of side lengths for each triangle is the same, and this is the case if and only if the sets of numbers of vertices in the n-gon between consecutive vertices of the triangles (including precisely one of the endpoints) are the same (we will not prove this). But the set of numbers of vertices of the n-gon between consecutive vertices of a triangle is precisely a partition of

n into 3 parts, and from what we have just said, different partitions precisely correspond to mutually incongruent triangles. So counting these mutually incongruent triangles is equivalent to counting the partitions of n into 3 parts.

We use the letters E, I, and S to denote the numbers of congruence classes of equilateral, isosceles, and scalene triangles respectively. Since any triangle falls into precisely one of these categories, we just need to show that $E + I + S = \{\frac{1}{12}n^2\}$.

Let us first consider equilateral triangles. An equilateral triangle corresponds to choosing 3 of the n vertices equally spaced on the n-gon, so that n must be a multiple of 3 in order for an equilateral triangle to be obtainable from the regular n-gon. On the other hand, if n is a multiple of 3, then we can obtain equilateral triangles from the vertices of the n-gon, but note that there is only one possible congruence class corresponding to an equilateral triangle, which corresponds to the partition $n/3 + n/3 + n/3$. Thus

$$E = 1_{3|n}.$$

Next, we count the total number of congruence classes of triangles with at least two sides of equal length (so we are computing $I + E$). We can count this number by starting from a fixed vertex v, and counting all of the triangles that can be obtained by choosing the remaining two points to be equidistant from v. If n is odd, there are $(n-1)/2$ ways of doing this; if n is even there are $n/2 - 1$ ways of doing this. In both cases, these triangles correspond to distinct congruence classes since the side lengths are different, and we can combine the formulas for the two cases into the single expression

$$I + E = \frac{n-1}{2} - \frac{1}{2} \cdot 1_{2|n}.$$

Finally, we consider all of the possible triangles we can obtain from the regular n-gon which contain a specific vertex v. The number of such triangles is precisely equal to $\binom{n-1}{2}$ since we need to choose two of the remaining vertices. Note that the collection of these triangles covers all congruence classes (since all other triangles can be obtained by rotation). However, it multiply counts triangles in the following way: if all edges have the same length, it only counts the congruence class once. For congruence classes of triangles with two edges the same length, the edge with different side length could occur in any of the three positions, so this counts isosceles triangles three times. Finally, for congruence classes of triangles with all three edges different lengths, this counts each of the possible permutations of the three edges, for a total of 6-fold counting of the scalene triangles. Thus, we have

$$E + 3I + 6S = \binom{n-1}{2} = \frac{n(n-1)}{2}.$$

We can combine the three formulas above to get

$$\begin{aligned} E+I+S &= \frac{1}{6}\left[(E+3I+6S)+3(E+I)+2E\right] \\ &= \frac{1}{6}\left[\frac{(n-1)(n-2)}{2}+3\frac{n-1}{2}-\frac{3}{2}\cdot 1_{2|n}+2\cdot 1_{3|n}\right] \\ &= \frac{1}{12}\left[n^2-3n+2+3(n-1)-3\cdot 1_{2|n}+4\cdot 1_{3|n}\right] \\ &= \frac{1}{12}\left[n^2-1-3\cdot 1_{2|n}+4\cdot 1_{3|n}\right]. \end{aligned}$$

Thus, we have

$$\left|\frac{1}{12}n^2-(E+I+S)\right| = \frac{1}{12}\left|-1-3\cdot 1_{2|n}+4\cdot 1_{3|n}\right| \leq 1/3,$$

and so $E+I+S=\left\{\frac{1}{12}n^2\right\}$, finishing the proof. $\square$