Math 108 Homework 5 Solutions

Problem 10H
Prove that for $0 \le k \le n$,

$$\sum_{i=0}^{k} \binom{k}{i} D_{n-i} = \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} (n-j)!.$$

Answer: Here we note that D_{n-i} represents the number of derangements of $n-i$ objects. I claim that both sides of this equation represent the number of permutations σ of $\{1, ..., n\}$ such that the fixed point set of σ (i.e., the set $\{x \mid \sigma(x) = x\}$) is a subset of $\{1, ..., k\}$.

First, to see that this is the case on the left hand side, note that for $A \subseteq \{1, ..., n\}$, the number of permutations with fixed point set A is equal to $D_{n-|A|}$, and we have

$$\begin{aligned} & \sum_{i=0}^{k} \binom{k}{i} D_{n-i} \\ = & \sum_{i=0}^{k} \sum_{A \subseteq \{1,...,k\} : |A| = i} D_{n-i} \\ = & \sum_{A \subseteq \{1,...,k\}} D_{n-|A|} \end{aligned}$$

is the total number of permutations for which the fixed point set is a subset of $\{1, ..., k\}$.

Next, we analyze the right hand side via inclusion-exclusion and Theorem 10.1. Let E_i be the set of those permutations with $\pi(i) = i$. By Theorem 10.1, the number of permutations which are not in E_i for any i with $k+1 \le i \le n$ is equal to

$$N - N_1 + N_2 - N_3 + ... + (-1)^{n-k} N_{n-k},$$

where $N = n!$ and $N_j := \sum_{|M|=j} |\cap_{i \in M} E_i|$, and the sum is over subsets M of $\{k+1, ..., n\}$. Note that we can compute $N_j = \sum_{|M|=j} (n-j)! = \binom{n-k}{j} (n-j)!$ since we are summing over subsets of the $(n-k)$-set $\{k+1, ..., n\}$. Thus, we have that the number of permutations not in E_i for any i with $k+1 \le i \le n$ is equal to

$$\sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} (n-j)!,$$

which is precisely the quantity on the right hand side. But note also that the statement that σ is not in E_i for any $i \in \{k+1, ..., n\}$ is precisely equivalent to the statement that the fixed point set of σ is a subset of $\{1, ..., k\}$. Thus, we have counted the same objects in two different ways, and so we have obtained the desired equality. □

Problem 14A

14A. (i). Let a_n denote the number of sequences of 0's and 1's that do not contain two consecutive 0's. Determine a_n.

Answer: Let s be an arbitrary sequence of length n which does not contain two consecutive 0's. Then s either ends with a 1, and the previous $n-1$ elements of the sequence can be any sequence length $n-1$ without two consecutive 0's (giving a_{n-1} possibilities), or it ends with a 0, and so the second to last element must be a 1, and the remaining $n-2$ elements can be any sequence of length $n-2$ without two consecutive 0's (giving an additional a_{n-2} possibilities). Thus, we have that $a_n = a_{n-1} + a_{n-2}$. Moreover, note that $a_0 = 1$ and $a_1 = 2$, and so the a_n satisfy the Fibonacci recursion with Fibonacci numbers for initial conditions.

We can solve this recursion using the power series approach from the book. If we set $f(x) = \sum_{n=0}^{\infty} a_n x^n$, we get

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= 1 + 2x + \sum_{n=2}^{\infty} a_n x^n \\ &= 1 + 2x + \sum_{n=2}^{\infty} (a_{n-1} + a_{n-2}) x^n \\ &= 1 + 2x + x \sum_{n=1}^{\infty} a_n x^n + x^2 \sum_{n=0}^{\infty} a_n x^n \\ &= 1 + 2x + x(f(x) - 1) + x^2 f(x) = 1 + x + (x + x^2) f(x), \end{aligned}$$

and so we have that $f(x) = \frac{1+x}{1-x-x^2}$.

Letting $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$ be the reciprocals of the roots of the denominator, we use the partial fraction decomposition to obtain

$$\frac{1+x}{1-x-x^2} = \frac{(1+\alpha)/(\alpha-\beta)}{1-\alpha x} + \frac{(1+\beta)/(\beta-\alpha)}{1-\beta x}.$$

From this, we get that

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{1+\alpha}{\alpha-\beta} \alpha^n + \frac{1+\beta}{\beta-\alpha} \beta^n \right) x^n,$$

and so

$$a_n = \frac{1+\alpha}{\alpha-\beta}\alpha^n + \frac{1+\beta}{\beta-\alpha}\beta^n.$$

14A. (ii). Let b_n denote the number of sequences of 0's and 1's with no two consecutive 1's and for which a run of 0's always has length 2 or 3, including possibly at the beginning or end of the sequence. Show that $b_n^{1/n} \to c$ for some c and approximate c.

Answer: First, we obtain a recursive formula for b_n. To do this, we introduce an auxiliary sequence c_n, where c_n is the number of such sequences ending in a 1. Note that any such sequence ending in a 1 has to have the final 1 preceded by either two zeros and a one, or three zeros and a one. Thus, we have that $c_n = c_{n-3} + c_{n-4}$. Next, note that $b_n = c_n + c_{n-2} + c_{n-3}$, where this summation corresponds to splitting the sequences into groups based on whether it ends with a 1, with two 0's, or with three 0's.

First, let's solve the recursion for c. Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$. Note that $c_0 = c_1 = c_3 = 1$ and $c_2 = 0$. Using this, we get that

$$\begin{aligned}
f(x) &= 1 + x + x^3 + \sum_{n=4}^{\infty} c_n x^n \\
&= 1 + x + x^3 + \sum_{n=4}^{\infty} (c_{n-3} + c_{n-4}) x^n \\
&= 1 + x + x^3 + x^3 \sum_{n=1}^{\infty} c_n x^n + x^4 \sum_{n=0}^{\infty} c_n x^n \\
&= 1 + x + x^3 + x^3(f(x) - 1) + x^4 f(x)
\end{aligned}$$

and so we have

$$f(x) = \frac{1+x}{1-x^3-x^4}.$$

By the partial fractions approach from examples in the book and the previous problem, it is clear from the fact that $f(x) = \frac{1+x}{1-x^3-x^4}$ that the power series expansion of f will have coefficients taking the form

$$c_n = \sum_{i=1}^{4} a_i \frac{1}{z_i^n},$$

where the a_i are some nonzero constants (nonzero since the expression for f does not reduce) and the z_i are the (complex) roots of the polynomial $1 - x^3 - x^4$ arranged in increasing order by magnitude. Since z_1 then has the smallest magnitude of all of the roots, we have that $c_n = \left(\frac{1}{z_1}\right)^n \sum_{i=1}^{4} a_i \left(\frac{z_1}{z_i}\right)^n$, and so $c_n^{(}1/n) \to 1/z_1$ as $n \to \infty$.

We may compute the roots of $1 - x^3 - x^4$ using the quartic formula, or by using a computer algebra program, and obtain the approximations $z_1 \approx 0.8192$, $z_2 \approx -0.219 - 0.914i$, $z_3 \approx -0.219 + 0.914i$, and $z_4 = -1.380$. As showed above, we have $c_n^{1/n} \to 1/z_1 \approx 1.2207$.

Finally, since $b_n = c_n + c_{n-2} + c_{n-3}$, we have that

$$c_n^{1/n} \leq b_n^{1/n} \leq \max_{i=0}^{3} (3c_{n-i})^{1/(n-i)},$$

but note that as $n \to \infty$ both the far left and far right sides of the above chain of inequalities tend to $1/z_1 \approx 1.2207$. Thus $b_n^{1/n} \to 1/z_1 \approx 1.2207$ as well. $\square$

Problem 14I

Let the points $1, ..., 2n$ be on a circle (consecutively). We wish to join them in pairs by n nonintersecting chords. In how many ways can this be done?

Answer: Let u_n be the Catalan numbers, so that $u_1 = 1$ and the u_n satisfy the recurrence

$$u_n = \sum_{m=1}^{n-1} u_m u_{n-m}.$$

Let c_n be the number of of ways of joining the points into chords in a circle with $2n$ points. We wil show that $c_n = u_{n+1}$.

First, note that $c_0 = 1 = u_1$ and $c_1 = 1 = u_2$. Second, note that we can reformulate the above recurrence for u_{n+1} as saying

$$u_{n+1} = \sum_{m=1}^{n} u_m u_{n+1-m} = \sum_{m=1}^{n} u_{m-1+1} u_{n-m+1}.$$

Thus, since we have agreement between the base cases c_1 and $u_{1+1} = u_2$, it is enough to show that c_n satisfies the recurrence

$$c_n = \sum_{m=1}^{n} c_{m-1} c_{n-m}.$$

Note that if we have a chord from point 1 to point i, then there can be no chord from any points in $\{2, ..., i-1\}$ to any point in $\{i+1, ..., 2n\}$ since that would intersect the chord from 1 to i. Thus, the number of ways of joining the points by nonintersecting chords such that point 1 is connected to point i is given by the product of the number of ways of joining points $\{2, ..., i-1\}$ and points $\{i+1, ..., 2n\}$ by nonintersecting chords. Note that in particular, if any such way of joining the points exists, then i must be even or else we will have two odd-sized sets, in which case no way of pairing up the points will exist.

Although these two subsets are no longer circular, the number of ways of joining them by nonintersecting chords is precisely the same as if they were circular. Thus, we can compute c_n by summing over the possible points that 1 can be adjacent to, and for each such point, counting the number of chord pairings with 1 adjacent to that point.

As we already noted, 1 must be paired with an even point, so let us assume that 1 is paired with $2j$. Then there are $2(j-1)$ points in $\{2, ..., 2j-1\}$, and $2(n-j)$ points in $\{2j+1, ..., 2n\}$. Thus, the number of nonintersecting chord pairings with points 1 and j paired is given by $c_{j-1}c_{n-j}$. Summing over the possible values of j, we obtain

$$c_n = \sum_{j=1}^{n} c_{j-1}c_{n-j}.$$

But this was precisely the recurrence we needed to show to prove that $c_m = u_{m+1}$ for all $m \geq 0$. $\square$

Problem 14N

Consider walks in the $X-Y$ plane where each step is $R : (x, y) \to (x+1, y)$, $U : (x, y) \to (x, y+1)$, or $D : (x, y) \to (x+1, y+1)$. We wish to count walks from $(0, 0)$ to (n, n) that are below or on the line $x = y$. Find a recurrence for the number of walks like (14.10).

Answer: Let b_n be the number of such walks below or on $x = y$, and let $b_{k,n}$ be the number of such walks from $(0, 0)$ to (n, n) below or on $x = y$ that meet the line $x = y$ for the second time in (k, k). Then we have that $b_n = \sum_{k=1}^{n} b_{k,n}$.

If we let c_k be the number of walks from $(0, 0)$ to (k, k) below the line $x = y$ which do not intersect the line $x = y$ between $(0, 0)$ and (k, k), and note that b_{n-k} is also equal to the number of walks on or below the line $x = y$ from (k, k) to (n, n), then it is clear that $b_{k,n} = c_k b_{n-k}$. Next, note that for $k > 1$, in order for a walk to be counted in c_k, it must start with an R step, and end with a U step, and must be a walk from $(1, 0)$ to $(k, k-1)$ on or below the line $x+1 = y$ in between. But the number of such walks is precisely equal to the number of walks from $(0, 0)$ to $(k-1, k-1)$ which are on or below $x = y$, which is precisely equal to b_{k-1}. Thus, we have that $c_k = b_{k-1}$ for $k > 1$. For $k = 1$, the presence of diagonal steps complicated things, and gives us $c_1 = 2 = b_0 + 1$. Thus, we have that

$$\begin{aligned} b_n &= \sum_{k=1}^{n} b_{k,n} \\ &= \sum_{k=1}^{n} c_k b_{n-k} \\ &= \left(\sum_{k=1}^{n} b_{k-1} b_{n-k} \right) + b_{n-1} \end{aligned}$$

where we obtained the b_{n-1} in the final line by using the fact that c_1 is equal to b_0+1. $\square$