Math 108 Homework 4 Solutions

Problem 10A
How many positive integers less than 1000 have no factor between 1 and 10?

Proof. Let p_i be the i^{th} prime number, so that $p_1 = 2$, $p_2 = 3$, etc. Let $E_i = \{n \in \mathbb{Z} \mid 1 \le n \le 1000 \text{ and } p_i \text{ divides } n\}$ for $1 \le i \le 4$. Noting that the intersection $\cap_{k=1}^{m} E_{i_k} = \{n \in \mathbb{Z} \mid 1 \le n \le 1000 \text{ and } \prod_{k=1}^{m} p_{i_k} \text{ divides } n\}$, and also noting that the size of the set $|\{n \in \mathbb{Z} \mid 1 \le n \le 1000 \text{ and } m \text{ divides } n\}|$ is equal to $\lfloor \frac{1000}{m} \rfloor$, we may apply Theorem 10.1 to obtain that the number of such positive integers is equal to

$$\begin{aligned} 1000 \quad &- \left(\left\lfloor\frac{1000}{2}\right\rfloor + \left\lfloor\frac{1000}{3}\right\rfloor + \left\lfloor\frac{1000}{5}\right\rfloor + \left\lfloor\frac{1000}{7}\right\rfloor\right) \\ &+ \left(\left\lfloor\frac{1000}{2\cdot 3}\right\rfloor + \left\lfloor\frac{1000}{2\cdot 5}\right\rfloor + \left\lfloor\frac{1000}{2\cdot 7}\right\rfloor + \left\lfloor\frac{1000}{3\cdot 5}\right\rfloor + \left\lfloor\frac{1000}{3\cdot 7}\right\rfloor + \left\lfloor\frac{1000}{5\cdot 7}\right\rfloor\right) \\ &- \left(\left\lfloor\frac{1000}{2\cdot 3\cdot 5}\right\rfloor + \left\lfloor\frac{1000}{2\cdot 3\cdot 7}\right\rfloor + \left\lfloor\frac{1000}{2\cdot 5\cdot 7}\right\rfloor + \left\lfloor\frac{1000}{3\cdot 5\cdot 7}\right\rfloor\right) \\ &+ \left(\left\lfloor\frac{1000}{2\cdot 3\cdot 5\cdot 7}\right\rfloor\right) = 228. \end{aligned}$$

□

Problem 13A
On a circular array with n positions, we wish to place the integers $1, 2, ..., r$ in order, clockwise, such that consecutive integers, including the pair $(r, 1)$, are not in adjacent positions on the array. Arrangements obtained by rotation are considered the same. In how many ways can this be done?

Proof. For $1 \le i < r$, let a_i denote the number of empty spaces between the position of i and the position of $i + 1$, and let a_r denote the number of empty spaces between the position of r and the position of 1. Since arrangements obtained by rotation are considered the same, the sequence $a_1, ..., a_r$ uniquely determines the arrangement. Note that there are a total of n spaces, r of which are filled by integers, for a total of $n - r$ empty spaces. Thus it must be the case that $\sum_i a_i = n - r$. The condition that the consecutive integers (and 1 and r) are not adjacent is equivalent to the condition that $a_i \ge 1$ for all i. This is the only constraint on the a_i's, and so the number of valid arrangements according to the requirements of the problem is equal to the number of solutions to $\sum_{i=1}^{r} a_i = n - r$ with $a_i \ge 1$. By the corollary to Theorem 13.1, this is equal to $\binom{n-r-1}{r-1}$. □

Problem 13B
Show that the following formula for binomial coefficients is a direct consequence of (10.6):

$$\binom{n+1}{a+b+1} = \sum_{k=0}^{n} \binom{k}{a}\binom{n-k}{b}.$$

Give a combinatorial proof by considering $(a+b+1)$-subsets of the set $\{0, 1, ..., n\}$, ordering them in increasing order, and then looking at the value of the integer in position $a+1$.

Proof. First, by applying the identity $\binom{n}{k} = \binom{n}{n-k}$ to each of the binomial coefficients appearing in the identity we wish to show, it is equivalent to show that

$$\binom{n+1}{n-a-b} = \sum_{k=0}^{n} \binom{k}{k-a}\binom{n-k}{n-k-b}.$$

By (10.6), we have that $\binom{k}{k-a}$ is the coefficient on x^{k-a} in $(1-x)^{-a-1}$, and $\binom{n-k}{n-k-b}$ is the coefficient on x^{n-k-b} in $(1-x)^{-b-1}$. If we consider the product of these two power series, we see that the coefficient on x^{n-a-b} in $(1-x)^{-a-b-2}$ is given by the sum of all products of coefficients of the power series $(1-x)^{-a-1}$ and $(1-x)^{-b-1}$ for which the sum of the degrees of the corresponding terms is equal to $n-a-b$. That is, the coefficient on x^{n-a-b} in $(1-x)^{-a-b-2}$ is equal to

$$\begin{aligned}
&\sum_{k=0}^{n-a-b} \binom{k+a}{k}\binom{n-a-b-k+b}{n-a-b-k} \\
=& \sum_{k=0}^{n-a-b} \binom{k+a}{k}\binom{n-a-k}{n-k-a-b} \\
=& \sum_{k=a}^{n-b} \binom{k}{k-a}\binom{n-k}{n-k-b} \\
=& \sum_{k=0}^{n} \binom{k}{k-a}\binom{n-k}{n-k-b}
\end{aligned}$$

where we obtained the final line by noting that all of the terms coming from values of k not included in the previous line are equal to zero. But we may also use (10.6) to obtain the formula for the coefficient on x^{n-a-b} in $(1-x)^{-a-b-2}$ directly, and get that it is equal to $\binom{n+1}{n-a-b}$. This finishes the proof of the desired identity using (10.6).

For the combinatorial proof, note that we may divide the set of all $(a+b+1)$-subsets of $\{1, ..., n+1\}$ in the following manner: let E_k be the set of all those subsets such that the $(a+1)^{\text{th}}$ largest element of the subset is equal to k. Then since every $(a+b+1)$-subset of $\{1, ..., n+1\}$ has an $(a+1)^{\text{th}}$ largest element, we have that

$$\binom{n+1}{a+b+1} = \sum_{k=1}^{n+1} |E_k| = \sum_{k=0}^{n} |E_{k+1}|.$$

But note that $|E_{k+1}|$ is the number of subsets of $\{1, ..., n+1\}$ of size $a+b+1$ containing $k+1$, and containing a elements smaller than $k+1$, and b elements larger than $k+1$. Thus, we have $|E_{k+1}| = \binom{k}{a}\binom{n-k}{b}$, since this is the number of ways of choosing such subsets. Thus, we have that

$$\binom{n+1}{a+b+1} = \sum_{k=0}^{n} \binom{k}{a}\binom{n-k}{b},$$

finishing the combinatorial proof. □

Problem 13E
The familiar relation

$$\sum_{m=k}^{\ell} \binom{m}{k} = \binom{\ell+1}{k+1}$$

is easily proved by induction. The reader who wishes to, can find a more complicated proof by using (10.6). Find a combinatorial proof by counting paths from $(0,0)$ to $(\ell+1, k+1)$ in the X-Y plane where each step is of type $(x,y) \to (x+1,y)$ or $(x,y) \to (x+1,y+1)$. Then use the formula to show that the number of solutions of

$$x_1 + x_2 + ... + x_k \leq n$$

in nonnegative integers is $\binom{n+k}{k}$. Can you prove this result combinatorially?

Proof. First, note that the number of paths from $(0,0)$ to (a,b) using only steps of the prescribed form is equal to $\binom{a}{b}$, since every step increases the x coordinate by 1, so that a steps must be taken, and during precisely b of these steps we must also increase the y coordinate by 1 (i.e., move diagonally), but beyond this there is no restriction on which steps are diagonal and which are horizontal. Thus, the number of paths from $(0,0)$ to $(\ell+1, k+1)$ of the prescribed form is equal to $\binom{\ell+1}{k+1}$ as desired.

To see that this is also given by the sum on the left side of the desired identity, we divide the space of paths from $(0,0)$ to $(\ell+1, k+1)$ into subsets in the following manner. Let E_j be the set of paths from $(0,0)$ to $(\ell+1, k+1)$ of the prescribed form such that the first point on the path with y coordinate $k+1$ has x coordinate equal to $j+1$. Then the E_i form a partition of the set of paths from $(0,0)$ to $(\ell+1, k+1)$ of the desired form. Note that all paths in E_j must reach $(j+1, k+1)$ by a diagonal move, so that they also pass through (j,k), and moreover that the set of paths in E_j is in bijective correspondence with the set of paths of the prescribed form that reach the point (j, k). As argued above, the number of such paths is equal to $\binom{j}{k}$, so that we have shown that

$$\binom{\ell+1}{k+1} = \sum_{m=0}^{\ell} |E_m| = \sum_{m=0}^{\ell} \binom{m}{k} = \sum_{m=k}^{\ell} \binom{m}{k}$$

noting that we obtained the final equality only by deleting terms which are equal to zero. Thus we have proved the desired identity.

Now, the number of solutions to $x_1 + x_2 + ... + x_k \leq n$ in nonnegative integers is equal to the sum of the number of solutions to $x_1 + x_2 + ... + x_k = m$ in nonnegative integers over all $m \leq n$. By Theorem 13.1, the number of such solutions is $\binom{m+k-1}{k-1}$. Thus we have that the number of solutions to $x_1 + x_2 + ... + x_k \leq n$ is equal to

$$\sum_{m=0}^{n} \binom{m+k-1}{k-1} = \sum_{m=k-1}^{n+k-1} \binom{m}{k-1} = \binom{n+k}{k},$$

where we obtained the final equality using the identity from the first part of the problem.

The last part of the problem is to find a combinatorial proof of this fact. Well, the number of solutions to $x_1 + ... + x_k \leq n$ in nonnegative integers is equal to the number of solutions to $x_1 + ... + x_k \leq n + k$ with $x_i \geq 1$. We can see this as the number of ways of creating $k + 1$ subsets of $n + k + 1$ balls (each of size at least one), where x_i corresponds to the size of the i^{th} subset (and so we ignore the size of the $(k + 1)^{\text{th}}$ subset, which is how we get an inequality). Since there are $n + k$ places to place the "sticks" between the $n + k + 1$ "balls" and obtain a division of the desired form, the number of ways of doing this is equal to $\binom{n+k}{k}$. Thus, the number of solutions to $x_1 + ... + x_k \leq n$ is equal to $\binom{n+k}{k}$.

□