Math 108 Homework 3 Solutions

Problem 3A
Fix an integer $d \geq 3$. Let H be a simple graph with all degrees $\leq d$ which cannot be d-colored and which is minimal (with the fewest vertices) subject to these properties. (We claim H is complete on $d+1$ vertices, but we dont know that yet.) (i) Show that H is nonseparable (this means that every graph obtained from H by deleting a vertex is connected). (ii) Then show that if the vertex set $V(H)$ is partitioned into sets X and Y with $|Y| \geq 3$, then there are at least three vertices $a, b, c \in Y$ each of which is adjacent to at least one vertex in X.

Proof.
(i) Suppose that H satisfies the stated properties, but is separable. Let v be a vertex such that removing $H \setminus \{v\}$ is not connected. Let H_1 be one of the connected components of $H \setminus \{v\}$, and let $H_2 = H \setminus (H_1 \cup \{v\})$. Since $H \setminus \{v\}$ is not connected, both H_1 and H_2 are nonempty, and so the induced subgraphs on $H_1 \cup \{v\}$ and $H_2 \cup \{v\}$ are proper subgraphs of H. Therefore, since H is minimal subject to the given properties, the induced subgraphs on both $H_1 \cup \{v\}$ and $H_2 \cup \{v\}$ can be d-colored. Note that permuting the colors in any valid coloring results in another valid coloring. Thus, if we take the coloring on $H_2 \cup \{v\}$ and permute the colors so that v has the same color in the H_2 coloring as the H_1 coloring, we may then combine these two colorings to give a d-coloring of all of H. This coloring on H is then a valid coloring because if v_1, v_2 are any two arbitrary vertices in H which are adjacent, then since there are no adjacencies between H_1 and H_2, it is either the case that both v_1 and v_2 are contained in $H_1 \cup \{v\}$, or they are both contained in $H_2 \cup \{v\}$. Since the coloring we have obtained on H was a valid coloring on each of these subsets individually, it is the case that the color of v_1 and v_2 is different in the combined coloring (on all of H), and since the combined coloring uses only d colors we have shown that H can be d-colored. This is a contradiction, and so we may conclude that H must be nonseparable.

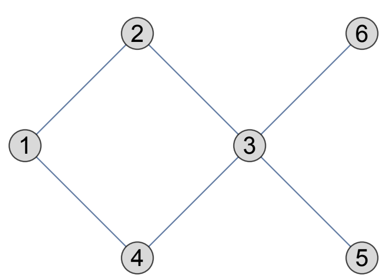
(ii) Suppose for the sake of contradiction that there is some partition of $V(H)$ into two nonempty sets X and Y with $|Y| \geq 3$ such that at most two vertices in Y are adjacent to at least one vertex in X. Note that if there were no vertices in Y adjacent to any vertex in X, then H would be disconnected, and so a d-coloring on each of its connected components (which are d-colorable since H is minimally non-d-colorable) could be combined to create a d-coloring of H, a contradiction. Also, if there was only one vertex in Y adjacent to a vertex in X, then H would be separable, since removing that vertex would leave no paths between X and the remaining vertices in Y. By the previous part, this would be a contradiction. Thus, there must be exactly two vertices $v_1, v_2 \in Y$ which are adjacent to a vertex in X. Note also that v_1 and v_2 must both be adjacent to some vertex in $Y \setminus \{v_1, v_2\}$, or the graph would be separable. To see this, suppose, for example, that v_1 was adjacent to no other vertex in Y – in that case, removing v_2 would leave the graph disconnected.

Now, consider the induced subgraph of H on Y. Since v_1 and v_2 are both adjacent to a vertex in X, they both have degree at most $d-1$ in this induced subgraph. Thus, if we let G_Y be the graph on Y which contains an edge between v_1 and v_2 along with all of the edges in the induced subgraph of H on Y, then G_Y has fewer vertices than H and all nodes still have degree less than or equal to d. Thus G_Y is d-colorable. Moreover, since v_1 and v_2 are adjacent, they take different colors under this coloring, and since G_Y contains all the edges that the induced subgraph of H on Y does, this is a d-coloring of Y. The same argument as above applied to $X \cup \{v_1, v_2\}$ instead of Y also gives a d-coloring of $X \cup \{v_1, v_2\}$ for which the colors of v_1 and v_2 are different. After this, the same color-permuting argument as was used in part (i) then allows us to combine these two colorings into a d-coloring of $X \cup Y = H$; we just permute the colors in one of the colorings so that v_1 has the same color in both colorings, and so does v_2 (this is possible since the color of v_1 is different than that of v_2 in both colorings). Thus, we have constructed a d-coloring of H, a contradiction. This concludes the proof. □

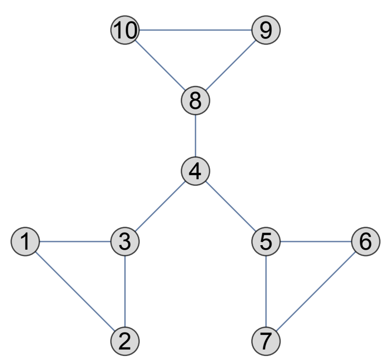
Handout 1

Compute the chromatic polynomial of the following graphs:

A



B



Computation for Graph A.

Recall the formula $\chi(G, k) = \chi(G-e, k) - \chi(G \circ e, k)$. For the graph A, we use this formula with the edge $e = (1, 4)$. Using this formula, we obtain that $\chi(A, k) = \chi(A - (1, 4), k) - \chi(A \circ (1, 4), k)$. Note that $A - (1, 4)$ is a tree on 6 vertices and so has chromatic polynomial $k(k-1)^5$. Next, we can compute the chromatic polynomial of $A \circ (1, 4)$ directly: $A \circ (1, 4)$ is isomorphic to a copy of K_3 with two additional vertices attached with one edge to one of the nodes in K_3. There are $k(k-1)(k-2)$ ways of coloring the copy of K_3, and then each of the remaining two nodes can be colored using any of the colors not already used by the node to which they are attached. This gives a total of $k(k-1)^3(k-2)$ colorings of $A \circ (1, 4)$. Thus, we have that

$$\begin{aligned}
\chi(G, k) &= k(k-1)^5 - k(k-1)^3(k-2) \\
&= k(k-1)^3((k-1)^2 - (k-2)) \\
&= k(k-1)^3(k^2 - 3k + 3)
\end{aligned}$$

□

Computation for Graph B.
We compute the chromatic polynomial $\chi(B, k)$ of this graph directly. We can choose the central vertex (vertex 4 in the picture above) arbitrarily, so that there are k possible choices of color for this vertex. Given a choice of color for this vertex, the color of each vertex in $\{3, 5, 8\}$ can be chosen arbitrarily among the remaining $k - 1$ colors. Once vertex 3 has been colored, vertices 1 and 2 can be colored using any two of the colors besides that of vertex 3, so that there are $(k-1)(k-2)$ possibilities for coloring vertices 1 and 2 given the color of 3. The analysis of the two remaining triangles is similar, and combining this all we see that the total number of k-colorings of the graph is $\chi(B, k) = k(k-1)^3((k-1)(k-2))^3 = k(k-1)^6(k-2)^3$. Just to clarify where the formula comes from, in $\chi(B, k) = k(k-1)^3((k-1)(k-2))^3$, the factor of k corresponds to the arbitrary choice of color for node 4, the $(k-1)^3$ term corresponds to the choices of color for vertices $3, 5$, and 8, and the $((k-1)(k-2))^3$ term corresponds to the choices of coloring for each of the three triangles. □