1) Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

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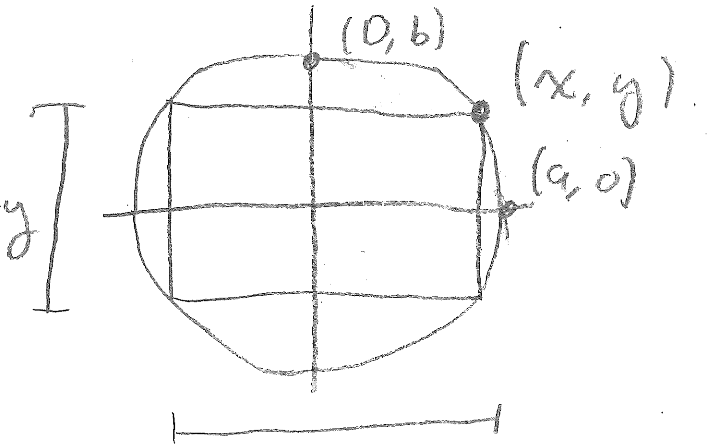
Picture:




Area: $A = (2x)\cdot(2y) = 4xy$.

Relation b/w $x$ & $y$: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\therefore \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$y = b\cdot\sqrt{1 - \frac{x^2}{a^2}}$$

Maximize $A = 4\,x\cdot b\sqrt{1 - \frac{x^2}{a^2}}$.

Trick: There's a $\sqrt{\ }$, so maximize

$$A^2 = 16b^2\,x^2\left(1 - \frac{x^2}{a^2}\right) = 16b^2x^2 - 16\frac{b^2}{a^2}x^4.$$

· Find critical points:

$$\frac{d}{dx}\left(A^2\right) = 32b^2x - 64\frac{b^2}{a^2}x^3$$

$$32b^2x - 64\frac{b^2}{a^2}x^3 = 0$$

$$\therefore\ 32b^2x\left(1 - 2\frac{x^2}{a^2}\right) = 0$$

$$\Rightarrow\ x = 0, \quad \text{or} \quad 1 - \frac{2}{a^2}x^2 = 0$$

$$\frac{2}{a^2}x^2 = 1 \rightsquigarrow x^2 = \frac{a^2}{2}$$

$$\Rightarrow x = \pm\sqrt{\frac{a^2}{2}}$$

$$= \pm\frac{a}{\sqrt{2}}$$

Domain of $x$: $[0, a]$ (from picture)

$\Rightarrow$ Check $x = 0, \frac{a}{\sqrt{2}}, a$

If $x = 0$, $A = 0$.

$x = a$, $A = 0$.

$x = \frac{a}{\sqrt{2}}$, $A = 4 \cdot \frac{a}{\sqrt{2}} \cdot b \cdot \sqrt{1 - \frac{a^2}{2a^2}}$

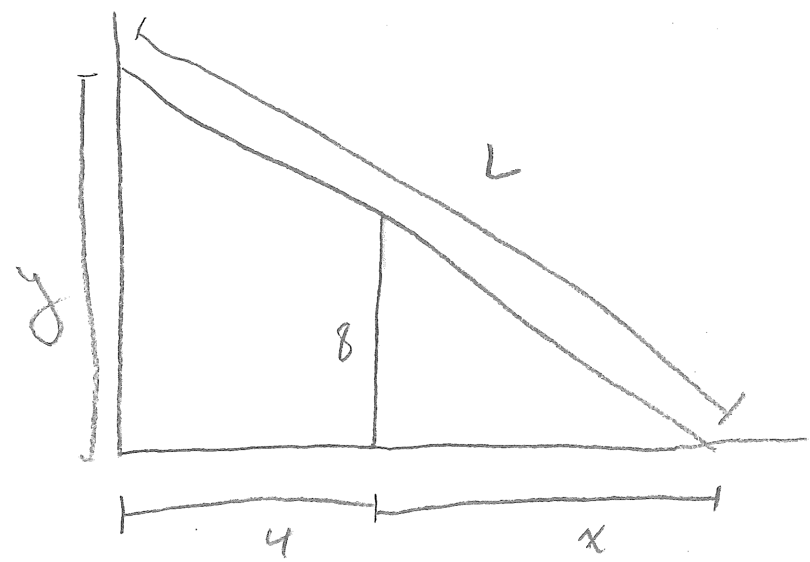
$= 2ab > 0$

Thus, $x = \frac{a}{\sqrt{2}}$ is the abs. max, a, in this case.

$$y = b\sqrt{1 - \frac{a^2}{2a^2}} = \frac{b}{\sqrt{2}} \qquad \boxed{A = 2ab}$$

2) A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft. Find the length of the shortest ladder that will reach over the fence to the building.

Picture:




$$L^2 = (x+4)^2 + y^2$$

Relation: $\frac{y}{x+4} = \frac{8}{x} \Rightarrow y = 8 \cdot \frac{x+4}{x}$

$$\Rightarrow L^2 = (x+4)^2 + \frac{8^2 (x+4)^2}{x^2}$$

$$= (x+4)^2 \left(1 + \frac{64}{x^2}\right)$$

$$\frac{dL^2}{dx} = (x+4)^2 \cdot \left(\frac{-128}{x^3}\right) + 2(x+4) \cdot \left(1 + \frac{64}{x^2}\right)$$

Solve: $(x+4)^2 \cdot \left(\frac{-128}{x^3}\right) + 2(x+4)\left(1 + \frac{64}{x^2}\right) = 0$

$$\frac{(x+4)}{x^3}\left((x+4)(-128) + 2\cdot\left(1+\frac{64}{x^2}\right)\cdot x^3\right) = 0$$

$x = -4$, or:

$$-128(x+4) + 2x^3\left(1+\frac{64}{x^2}\right) = 0$$

$$-\cancel{128x} - 512 + 2x^3 + \cancel{128x} = 0$$

$$2x^3 = 512$$

$$x^3 = 256$$

$$x = \sqrt[3]{256} \ldots$$

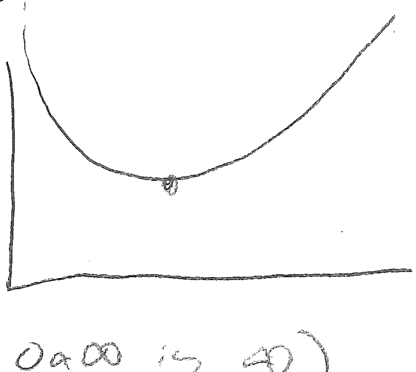
This is probably the answer. Let's check!

Domain of $x$: $(0, \infty)$.

Not a closed interval. But there's only one critical point, so if it's a local max or min, it's an absolute max or min. (Think about this.)

As $x \to 0$ or $\infty$, $L \to \infty$. (Think about this, too).

Graph of $L$:



$\Rightarrow x = \sqrt[3]{256}$ is a abs. min

& $L = \sqrt{(x+4)^2\left(1+\frac{64}{x^2}\right)}$

(one CP. limit at $0$ & $\infty$ is $\infty$).

3) A boat leaves a dock at 2:00 PM and travels due south at 20 km/h. Another boat has been traveling due east and reaches the same dock at 3:00 PM. At what time were the two boats closest together?

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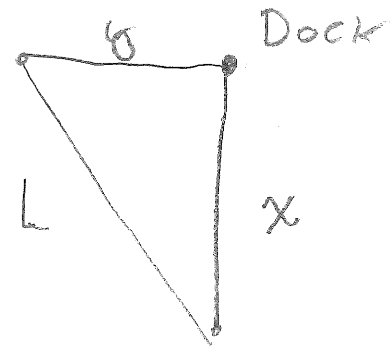
Picture:




$$L^2 = x^2 + y^2.$$

We want a time for our answer: express $x$ & $y$ in terms of $t$.

$x = 20 \cdot t$ (dist. = rate · time). ($t$ is in hours)

For $y$, we note at 2:00 PM, it was one hour away. Thus, initially, it must have been

$15 \frac{km}{h} \cdot 1\,h = 15$ km away.

It's getting closer to the dock, so $y = 15 - 15t$.

Thus, $L^2 = (20t)^2 + (15-15t)^2$

$$= 400t^2 + 225 - 450t + 225t^2$$

$$= 625t^2 - 450t + 225.$$

$$\frac{dL^2}{dt} = 1250t - 450.$$

So CP @ $1250t = 450.$

$$t = \frac{450}{1250} = \frac{9}{25}.$$

Is this the minimum? let's check.

Domain of $t$: $[0, 1]$.

$$L\Big|_{t=0} = \sqrt{225} = 15 \quad \text{(we knew this, actually)}$$

$$L\Big|_{t=1} = \sqrt{400} = 20 \quad \text{(this, too)}.$$

$$L\Big|_{t=9/25} = \sqrt{625 \cdot \frac{81}{625} - 450 \cdot \frac{9}{25} + 225}$$

$$= \sqrt{81 - 162 + 225}$$

$$= \sqrt{144} = 12.$$

So $t = 9/25$ is the absolute minimum.

4) A steel pipe is being carried down a hallway that is 9 ft wide. At the end of the hall, there is a right angled turn into a narrower hallway that is 6ft wide. What is the longest pipe that can be carried horizontally through the corner?

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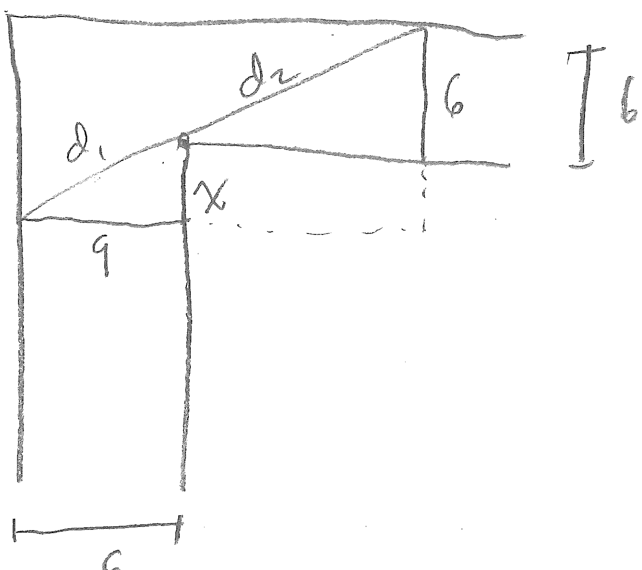
Picture:




Notice: the two triangles are similar. Thus,

$$\frac{d_2}{d_1} = \frac{6}{x} \Rightarrow d_2 = \frac{6}{x} d_1.$$

$$d_1 = \sqrt{x^2+81}$$

Minimize $D = d_1 + d_2 = d_1 + \frac{6}{x} d_1 = d_1\left(1+\frac{6}{x}\right) = \sqrt{x^2+81}\left(1+\frac{6}{x}\right)$

$$D^2 = (x^2+81)\left(1+\frac{6}{x}\right)^2.$$

$$\frac{dD^2}{dx} = (x^2+81)\cdot 2\left(1+\frac{6}{x}\right)\cdot\left(\frac{-6}{x^2}\right) + 2x\cdot\left(1+\frac{6}{x}\right)^2$$

$$= \left(1+\frac{6}{x}\right)\left(\frac{-12}{x^2}(x^2+81) + 2x\left(1+\frac{6}{x}\right)\right).$$

so: either $1+\frac{6}{x}=0$ ($\rightsquigarrow x=-6$, non sense)

or $0=\frac{-12}{x^2}(x^2+81)+2x\left(1+\frac{6}{x}\right)$.

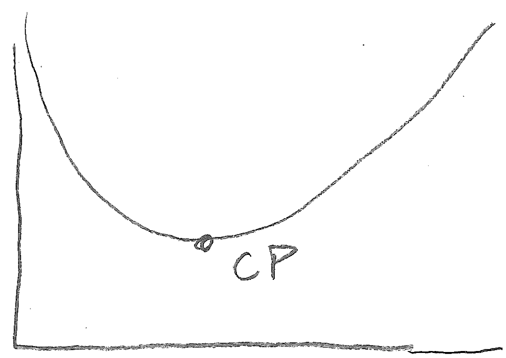
$$=\cancel{-12}-\frac{81\cdot 12}{x^2}+2x+\cancel{12}$$

$$=-\frac{81\cdot 12}{x^2}+2x$$

$$\Rightarrow \frac{81\cdot 12}{x^2}=2x \rightsquigarrow 81\cdot 6=x^3 \rightsquigarrow x=\sqrt[3]{486}.$$

Domain of $x$: $(0,\infty)$.

As in 2), there's only one CP, and as $x\to 0$ or $\infty$, $D\to\infty$. Thus, the graph of $D$ must be something like




so $x_0=\sqrt[3]{486}$ is the minimum.

$\Rightarrow D=\sqrt{x_0^2+81}\left(1+\frac{6}{x_0}\right)$ is the minimum distance, hence it's also the maximum length of pipe.