Bounds on the number of number fields of given degree and bounded discriminant

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(joint w/ Frank Thorne)

Preprint: https://arxiv.org/abs/2005.14110
Slides: https://rlemke01.math.tufts.edu/slides/nf-bounds.pdf

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Best known upper bound: $O(X^2)$

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Naive thought: If it's easy to write down irreducible polynomials, shouldn't it be easy to write down number fields?

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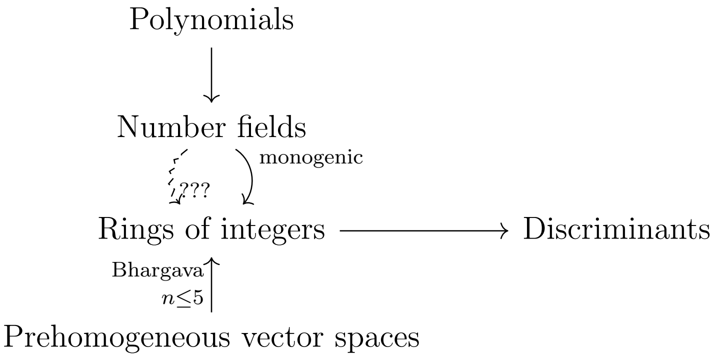
Example: $K = \mathbb{Q}[x]/(x^3 + 4x^2 + 3x + 8)$ needs $m = 2$

A schematic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???
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Discriminants
Bhargava
$n \leq 5$
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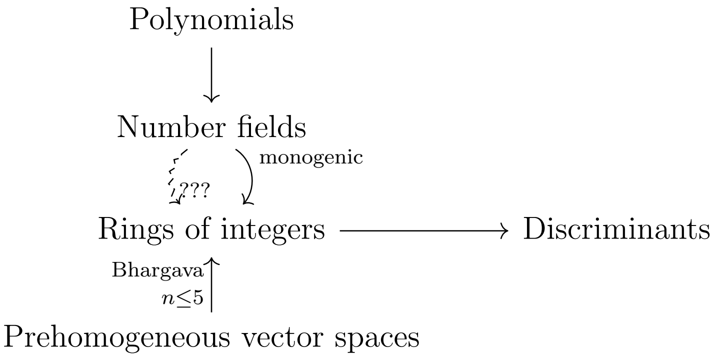

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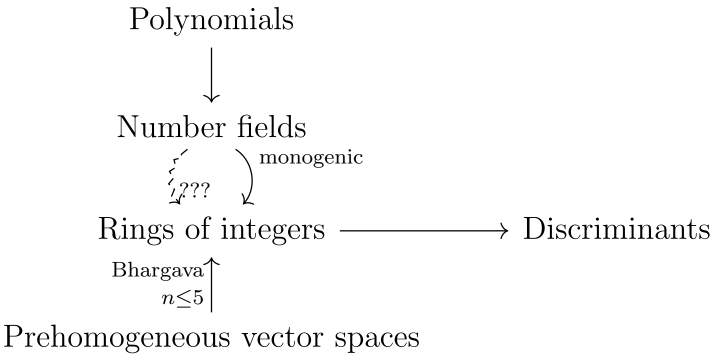

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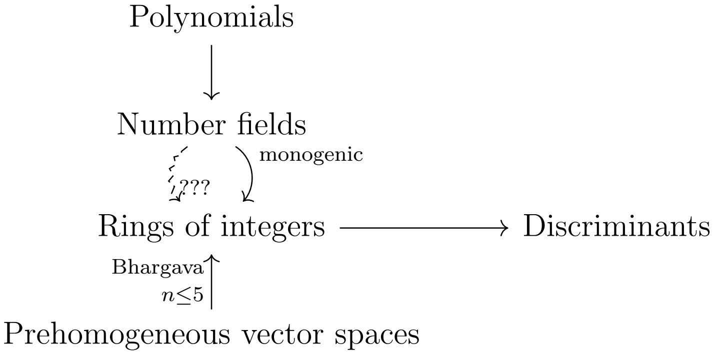

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Upshot: We have to settle for upper and lower bounds when $n \geq 6$

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Previous work of Schmidt, Ellenberg–Venkatesh, Couveignes.

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- **AIM** (2022+; in progress): Improve Schmidt for all n
  - Lose to LO–Thorne for n sufficiently large (e.g., $n \geq 100$)

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Schmidt's Idea, pt. 2

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In fact, $\exists \alpha \in \mathcal{O}_K$ with $||\alpha|| \ll_n |\mathrm{Disc}(K)|^{\frac{1}{2n-2}}$ and $\mathrm{Tr}_{K/\mathbb{Q}}(\alpha) = 0$.

Then $f_\alpha(x) = x^n + a_2 x^{n-2} + \cdots + a_n$, with

$$a_i \ll_n |\mathrm{Disc}(K)|^{\frac{i}{2n-2}} \leq X^{\frac{i}{2n-2}}.$$

There are $\ll_n X^{\frac{2}{2n-2} + \cdots + \frac{n}{2n-2}} = X^{\frac{n+2}{4}}$ such polynomials in $\mathbb{Z}[x]$,
$\Rightarrow$ there are $\ll_n X^{\frac{n+2}{4}}$ fields.

This is Schmidt's theorem.

(**Caution:** Slight issue: what if $K \neq \mathbb{Q}(\alpha)$? Schmidt inducts, details not important for this talk.)

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Good idea: Let α and β mingle. Consider $\mathrm{Tr}_{K/\mathbb{Q}}(\alpha^i \beta^j) \in \mathbb{Z}$.

Invariants of pairs α, β

Suppose $\alpha, \beta \in \mathcal{O}_K$. Then

$$\mathrm{Tr}_{K/\mathbb{Q}}(\alpha^i \beta^j) = \alpha_1^i \beta_1^j + \alpha_2^i \beta_2^j + \cdots + \alpha_n^i \beta_n^j.$$

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Theorem: $N_n(X) \ll_n X^{\frac{8}{3}\sqrt{n}}$.

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Question: How do we actually show a set of traces is enough?

Enough is enough: an example

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$$\begin{aligned} T_{1,0}\colon x_1 + x_2 + x_3 &= \mathrm{Tr}(\alpha), & T_{0,1}\colon y_1 + y_2 + y_3 &= \mathrm{Tr}(\beta), \\ T_{2,0}\colon x_1^2 + x_2^2 + x_3^2 &= \mathrm{Tr}(\alpha^2), & T_{1,1}\colon x_1 y_1 + x_2 y_2 + x_3 y_3 &= \mathrm{Tr}(\alpha\beta), \\ T_{0,2}\colon y_1^2 + y_2^2 + y_3^2 &= \mathrm{Tr}(\beta^2), & T_{3,0}\colon x_1^3 + x_2^3 + x_3^3 &= \mathrm{Tr}(\alpha^3). \end{aligned}$$

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Actual goal: Want to show the variety cut out by these eq'ns has dimension 0.

Computing dimensions

Goal: Show that dim $V\,(T_{1,0}, T_{0,1}, T_{2,0}, T_{1,1}, T_{0,2}, T_{3,0}) = 0$.

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Compute the tangent space, i.e. the kernel of the 6×6 matrix

$$D := \begin{pmatrix} \nabla T_{1,0} \\ \nabla T_{0,1} \\ \nabla T_{2,0} \\ \nabla T_{1,1} \\ \nabla T_{0,2} \\ \nabla T_{3,0} \end{pmatrix}.$$

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$$\det D = -12(x_1-x_2)(x_1-x_3)(x_2-x_3)(x_1y_2-x_1y_3-x_2y_1+x_2y_3+x_3y_1-x_3y_2).$$

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Lemma

If $P\colon (\mathbb{C}^n)^r \to \mathbb{C}$ *is a non-zero polynomial and* $[K : \mathbb{Q}] = n$*,*

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If $P\colon (\mathbb{C}^n)^r \to \mathbb{C}$ is a non-zero polynomial and $[K : \mathbb{Q}] = n$, then there exist $\alpha_1, \ldots, \alpha_r \in \mathcal{O}_K$ with each $||\alpha_i|| \ll_{n,P} |\mathrm{Disc}(K)|^{1/n}$ such that $P(\vec{\alpha_1}, \ldots, \vec{\alpha_r}) \neq 0$.

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Applied to $\det D$ with $n = 3$, $r = 2$, we find:

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In general, we've transformed the problem into showing a (horrible!) determinant is a non-zero polynomial.

Determinants when $r = 2$

Theorem (LO–Thorne; $r = 2$)

If D is the $2n \times 2n$ matrix of partial derivatives of the first $2n$ functions $T_{1,0}, T_{0,1}, T_{2,0}, T_{1,1}, \ldots$, with

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Proof.

Induction. $n \mapsto n + 1$ gives two new rows and two new columns.
Cofactor expansion $\Rightarrow$ new 2×2 contribution not canceled. □

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If D is the $2n \times 2n$ matrix of partial derivatives of the first $2n$ functions $T_{1,0}, T_{0,1}, T_{2,0}, T_{1,1}, \ldots$, with

$$T_{a,b} := \sum_{i=1}^{n} x_i^a y_i^b,$$

then $\det D$ *is a non-zero polynomial in $x_1, \ldots, y_n$.*

Proof.

Induction. $n \mapsto n+1$ gives two new rows and two new columns. Cofactor expansion $\Rightarrow$ new 2×2 contribution not canceled. □

Leads to the bound $N_n(X) \ll X^{\frac{8}{3}\sqrt{n}}$.

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Summary

Theorem (LO–Thorne; explicit version)

1) Let d be the least integer for which $\binom{d+2}{2} \geq 2n+1$. *Then*

$$N_n(X) \ll_n X^{2d - \frac{d(d-1)(d+4)}{6n}} \ll X^{\frac{8\sqrt{n}}{3}}.$$

2) Let $3 \leq r \leq n$ *and let d be such that* $\binom{d+r-1}{r-1} \geq rn$. *Then* $N_n(X) \ll_{n,r,d} X^{dr}$.

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Theorem (LO–Thorne; asymptotic version)

There is a constant $c > 0$ such that $N_n(X) \ll_n X^{c(\log n)^2}$. In fact, $c = 1.564$ is admissible.

Thank you!