

REPRESENTATION BY TERNARY QUADRATIC FORMS

ROBERT J. LEMKE OLIVER

ABSTRACT. The problem of determining when an integral quadratic form represents every positive integer has received much attention in recent years, culminating in the 15 and 290 Theorems of Bhargava-Conway-Schneeberger and Bhargava-Hanke. For ternary quadratic forms, there are always local obstructions, but one may ask whether there are ternary quadratic forms which represent every locally represented integer. Indeed, such forms exist and are called regular, and Jagy, Kaplansky, and Schiemann proved that there are at most 913; however, only 899 of these are actually known to be regular. We consider the remaining 14 forms, and establish the regularity of each under the generalized Riemann Hypothesis, following the method pioneered by Ono and Soundararajan. Moreover, we consider the exceptional arithmetic consequences if a large, locally represented integer is not globally represented by a ternary quadratic form, proving that some Dirichlet L -function would necessarily have a Siegel zero or that some quadratic twist of an elliptic curve would have an unusually large Tate-Shafarevich group.

1. INTRODUCTION AND STATEMENT OF RESULTS

The problem of determining, given a positive definite integral quadratic form, the integers represented by the quadratic form has motivated, and indeed encodes, a great deal of modern number theory. The problem of determining which forms are *universal* – forms that represent every positive integer – originates with Lagrange’s four squares theorem, but it is only recently that a complete characterization has been found; this is the so called “15 Theorem” of Bhargava, Conway, and Schneeberger [1] and the “290 Theorem” of Bhargava and Hanke [2]. In addition, there is very recent work of Rouse [17] proving a “451 Theorem” for representation of odd integers.

When dealing with such problems, arguably the deepest case is that of ternary quadratic forms, bearing in mind that there are always local obstructions, so that the interesting problem becomes to determine the locally represented integers which are globally represented. The reason for the depth in this case is that the number of representations of an integer can be canonically decomposed into a “large” part and a “small” part, neither of which is well-understood. These notions are only valid asymptotically, and a theorem on representations follows by determining the point after which the large part truly is larger than the small part, a method first explicitly employed by Ono and Soundararajan [16] in their study of Ramanujan’s ternary quadratic form; these techniques have subsequently been improved upon by Kane [14], Jetchev and Kane [13], and Chandee [5]. Both the large and small parts have an alternative arithmetic interpretation: the large part corresponds to the class number of an imaginary quadratic field (and hence to the value of a Dirichlet L -function, which can be ineffectively bounded from below by Siegel’s theorem), and the small part corresponds to the central critical value of a modular L -function. Thus, the general problem of determining

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when the large part dominates requires a great deal of understanding of the behavior of L -functions, much of which is beyond our current technology.

Here, we are concerned principally with classifying *regular* positive definite integral ternary quadratic forms. A quadratic form is regular if the only obstructions to representation are local obstructions, which, as mentioned above, is the natural generalization of universal forms to the ternary setting. Jagy, Kaplansky, and Schiemann [12] proved that there are at most 913 regular ternary quadratic forms, and they proved that 891 of these forms are indeed regular. In subsequent work of Oh [15], eight more forms in the list of 913 were proved to be regular. The purpose of this paper is to establish that the remaining 14 forms are regular, albeit conditionally; see Sections 2.2 and 2.3 for the list of these forms.

Theorem 1.1. *Assume the GRH for all Dirichlet L -functions and all modular L -functions. Then each of the remaining 14 ternary quadratic forms mentioned above is regular.*

Remark. As the proof of Theorem 1.1 will show, we do not actually need the GRH for all modular L -functions. Rather, we need it for the set of quadratic twists of certain weight two newforms.

While it is obviously unfortunate that we are not able to provide an unconditional proof of this result, the fact that the GRH plays a role should not be surprising. The only general method to obtain results on representation depends on the decomposition into the large and small parts mentioned above, both of which encode values of L -functions. We understand both objects very well assuming the GRH, but for neither do we currently possess unconditional bounds of sufficient quality. We remark that, to the author's knowledge, no set of integers represented by a positive definite ternary quadratic form which does not globally represent at least two locally represented integers has been determined without assuming GRH.

Motivated by work of Granville and Stark [9], who established that a form of the *abc*-conjecture implies that there are no Siegel zeros of Dirichlet L -functions, we consider what exceptional arithmetic consequences would arise from the failure of a large locally represented integer to be globally represented.

Theorem 1.2. *Let Q be a ternary quadratic form of discriminant Δ , and assume the GRH for the family of L -functions associated to quadratic twists of newforms of conductor dividing Δ . Moreover, given any integer n , let \bar{n} denote the image of n in the finite set $\prod_{p|\Delta} \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2$. Then there is an explicitly computable constant $N(Q, \mathbf{a})$ such that if $n \geq N(Q, \mathbf{a})$, $\bar{n} = \mathbf{a}$, n is squarefree, and n is locally represented but is not globally represented, then there is a Siegel zero of some Dirichlet L -function.*

Remark. By a Siegel zero of a Dirichlet L -function, we mean a real zero $\sigma < 1$ of some $L(s, \chi)$, where χ is a primitive real Dirichlet character to the modulus q and

$$\sigma > 1 - \frac{c}{\log 3q},$$

where c is some positive real number. Of course, we allow the quantity $N(Q, \mathbf{a})$ in Theorem 1.2 to depend upon the choice of c .

The constant $N(Q, \mathbf{a})$ in Theorem 1.2 is especially nice in the case that the cuspidal part of the theta function associated to Q is a Hecke eigenform. As an example of this, we have the following application to Ramanujan's ternary quadratic form, $Q = x^2 + y^2 + 10z^2$, which,

in their pioneering paper, Ono and Soundararajan [16] proved represents all odd integers greater than 2719 under the assumption of the GRH.

Corollary 1.1. *Assume the GRH for the L -functions of all quadratic twists of the elliptic curve $y^2 = x^3 + x^2 + 4x + 4$. If the quadratic form $Q = x^2 + y^2 + 10z^2$ does not represent an odd integer $n \geq 2.8 \cdot 10^{25}$, then some Dirichlet L -function has a Siegel zero with*

$$c = 342395 \cdot n^{-0.392} \log^2 n.$$

Moreover, if c is fixed, if Q does not represent a locally represented integer $n \geq 8.179 \cdot 10^{24} \cdot c^{-2.793}$, then some Dirichlet L -function possesses a Siegel zero.

Lastly, for completeness, we consider the complementary question of, assuming the GRH for Dirichlet L -functions, deducing from the failure of a locally represented integer to be globally represented exceptional behavior for the arithmetic of modular forms. While we are able to state a more general result (see the third remark following the theorem), we focus on a case of more arithmetic interest. We say that a ternary quadratic form Q is *associated to an elliptic curve* E/\mathbb{Q} if the cuspidal part of its theta function is a Hecke eigenform which lifts, under the Shimura correspondence, to the cusp form associated to E . Also, given any elliptic curve E , let E_d denote its quadratic twist by d , and let $\mathbb{III}(E_d)$ denote the Tate-Shafarevich group of E_d .

Theorem 1.3. *Assume the GRH for Dirichlet L -functions and the Birch and Swinnerton-Dyer conjecture for rank 0 elliptic curves, and suppose that Q is associated to the elliptic curve E . If n is locally represented but not globally, then there is a positive integer $d \gg n$ for which*

$$|\mathbb{III}(E_{-d})| \gg_E \frac{d}{\log^4 d},$$

where the implied constant can be made explicit.

Three remarks:

1) Ramanujan's quadratic form $x^2 + y^2 + 10z^2$ is an example of a quadratic form associated to an elliptic curve; namely, it is associated to the curve $y^2 = x^3 + x^2 + 4x + 4$ given in Corollary 1.1. In this case, we would have $d = 10n$.

2) While the lower bound on the size of $\mathbb{III}(E_d)$ does not contradict the Goldfeld-Szpiro conjecture [7] that, for any E/\mathbb{Q} with conductor N , $|\mathbb{III}(E)| \ll_\epsilon N^{1/2+\epsilon}$ uniformly in E , in fact a stronger statement is expected for the family of quadratic twists. In particular, the Ramanujan conjecture for half-integral weight modular forms would imply that $|\mathbb{III}(E_d)| \ll_{E,\epsilon} d^{1/2+\epsilon}$.

3) In the event that Q is not associated to an elliptic curve, it is still possible to deduce similar sorts of arithmetic implications. To any packet of Galois representations, and in particular to a newform, one can associate a Tate-Shafarevich group, and it is possible, under the appropriate conjectures, to deduce that some quadratic twist of a newform associated to Q would have an usually large Tate-Shafarevich group in this sense. See, for example, work of Bloch and Kato [3] for more information on such objects.

This paper is organized as follows. In Section 2, we go over the necessary background in more detail, and we prove Theorem 1.1. In Sections 3 and 4, we prove Theorems 1.2 and 1.3, respectively.

2. REPRESENTATION BY TERNARY QUADRATIC FORMS

We begin this section by going into more detail on the decomposition alluded to in the introduction, and we discuss the general approach to be taken to prove Theorem 1.1; this comprises Section 2.1. This approach proves to be technically slightly easier in the case that the form in question is in a genus of size two. This is the case for 11 of the 14 forms, and we prove that each is regular in Section 2.2. The remaining three forms are each in a genus of size three, and we dispatch these in Section 2.3.

2.1. Eisenstein series and cusp forms. We begin with a brief review of the theory of quadratic forms as it relates to the theory of modular forms. Since we are only concerned with the case of positive definite integral ternary quadratic forms, it is to be understood that when we talk about a quadratic form, it is assumed to be such. Now, given two quadratic forms Q_1 and Q_2 , we say that Q_1 and Q_2 are (globally) equivalent if there is some matrix $\gamma \in GL_3(\mathbb{Z})$ such that the variable substitution $(x, y, z) \mapsto \gamma \cdot (x, y, z)$ takes Q_1 to Q_2 ; we say that they are locally equivalent if for each prime p , there is some matrix $\gamma_p \in GL_3(\mathbb{Z}_p)$ taking Q_1 to Q_2 . The genus of a form Q , denoted by $\mathcal{G}(Q)$, is the set of forms locally equivalent to Q modulo global equivalence.

We can express Q in the form

$$Q(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T A \mathbf{x},$$

where A is a symmetric 3×3 matrix with integer entries and even diagonal entries. The discriminant Δ of Q is the determinant of A , and the level of Q is the least integer N for which NA^{-1} has integer entries and even diagonal entries. The theta function associated to Q is given by

$$\theta_Q(z) := \sum_{\mathbf{x} \in \mathbb{Z}^3} q^{Q(\mathbf{x})}, \quad q := e^{2\pi iz},$$

and it is a classical fact that $\theta_Q(z)$ is a modular form of weight $3/2$, level N , and nebentypus $\left(\frac{-4\Delta}{\cdot}\right)$. As such, it can be decomposed as

$$(2.1) \quad \theta_Q(z) = E(z) + C(z),$$

where $E(z)$ is an Eisenstein series and $C(z)$ is a cusp form. In fact, $E(z)$ can always be found from the theta functions of the forms in the genus of Q by the formula

$$E(z) = \frac{\sum_{Q' \in \mathcal{G}(Q)} \frac{1}{|\text{Aut}(Q')|} \theta_{Q'}(z)}{\sum_{Q' \in \mathcal{G}(Q)} \frac{1}{|\text{Aut}(Q')|}},$$

where $\text{Aut}(Q')$ denotes the (finite) automorphism group of Q' . In addition, there is the Siegel mass formula, which asserts that the Fourier coefficients of $E(z)$ are essentially class numbers of imaginary quadratic fields multiplied by certain local densities. In particular, we have, if $E(z) = \sum a_E(n)q^n$, that

$$(2.2) \quad a_E(n) = \frac{24h(-Mn)}{Mw(-Mn)} \prod_{p|2N} \beta_p(n) \cdot \frac{1 - \chi(p) \left(\frac{n}{p}\right) p^{-1}}{1 - p^{-2}},$$

where $h(-d)$ denotes the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, $w(-d)$ denotes the number of roots of unity in $\mathbb{Q}(\sqrt{-d})$, M is a rational number depending on

$n \pmod{8N^2}$ such that $-nM$ is a fundamental discriminant, and the quantities $\beta_p(n)$ are certain local densities depending on the image of n in the finite set

$$\prod_{p|\Delta} \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2.$$

See work of Hanke [11] or Rouse [17] for more information on these densities. Hence, if n is locally represented, each $\beta_p(n) \neq 0$, and we have that

$$a_E(n) \gg_Q h(-Mn) \gg_\epsilon n^{1/2-\epsilon},$$

where the last inequality is the notoriously ineffective theorem of Siegel. Currently, the best effective lower bound on class numbers is due to Goldfeld [8]. This relies on the deep work of Gross and Zagier [10], and would yield $a_E(n) \gg_\epsilon \log^{1-\epsilon} n$. As we will see, this bound is far too small to be of use, and it is for this reason that we must assume the GRH for Dirichlet L -functions. Assuming the GRH, the best explicit lower bounds on class numbers are due to Chandee [5], and are, of course, of the same quality as Siegel's lower bound.

To handle the cuspidal part $C(z)$ of $\theta_Q(z)$, we note that $C(z)$ can be decomposed as a linear combination of eigenforms. Each of these is either a unary theta function, which necessarily has coefficients supported on a single square class, or is subject to a theorem of Waldspurger [18], which says that the coefficients in certain square classes are essentially the square roots of central L -values of quadratic twists of the integral weight cusp form associated to the eigenform via Shimura's correspondence. Precisely, we have the following.

Theorem (Waldspurger). *Suppose that $f(z) \in S_{\lambda+1/2}(\Gamma_0(N), \chi)$ is a half-integer weight eigenform of each of the Hecke operators $T(p^2)$, $p \nmid N$, and eigenvalues $\lambda(p)$. Moreover, assume that $F(z) \in S_{2\lambda}(\Gamma_0(N), \chi^2)$ has the same system of eigenvalues under each $T(p)$. If n_1 and n_2 are two positive squarefree integers with $n_1/n_2 \in (\mathbb{Q}_p^\times)^2$ for each $p \mid N$ and $f(z) = \sum_{n=1}^{\infty} a(n)q^n$, then*

$$a(n_1)^2 n_2^{\lambda-1/2} \chi\left(\frac{n_2}{n_1}\right) L(1, F \otimes \chi^{-1} \chi_{n_2(-1)^\lambda}) = a(n_2)^2 n_1^{\lambda-1/2} L(1, F \otimes \chi^{-1} \chi_{n_1(-1)^\lambda}).$$

Now, assuming either the Ramanujan conjecture for half-integral weight cusp forms or the GRH for the family of quadratic twists of weight two newforms, it is possible to show that the Fourier coefficients $a_C(n)$ satisfy $a_C(n) \ll_\epsilon n^{1/4+\epsilon}$. Unconditionally, the convexity bound for the family of quadratic twists yields that $a_C(n) \ll_\epsilon n^{1/2+\epsilon}$, which is not sufficient to establish any asymptotic result (recall that $a_E(n) \gg_\epsilon n^{1/2-\epsilon}$). The best known subconvexity bound, due to Blomer and Harcos [4], yields that $a_C(n) \ll_\epsilon n^{7/16+\epsilon}$, which, combined with Siegel's theorem, is enough to establish an asymptotic result. Blomer and Harcos's result is effective, but, even assuming the GRH for Dirichlet L -functions (so as to obtain an effective lower bound on $a_E(n)$), the bound is not strong enough to yield a computationally feasible problem. Thus, in order to obtain an effective and usable result, we must assume the GRH. Doing so, the best explicit results for the cuspidal coefficients are again due to Chandee [5].

At this stage, assuming the GRH for Dirichlet L -functions and for the family of L -functions associated to quadratic twists of certain weight two newforms, we are able to obtain that if n is locally represented but is not globally represented, then $a_E(n) + a_C(n) = 0$, and from Chandee's bounds, we are able to rule this out for large values of n (strictly speaking, one must ensure that $a_C(n)$ has no contribution from a unary theta function, which can be done, e.g., by assuming that the squarefree part of n is large; however, in all of the cases of

interest to us in Theorem 1.1, it will turn out that there is no contribution from unary theta functions). A finite computation then suffices to establish the result. Unsurprisingly, if the cuspidal part of $\theta_Q(z)$ is an eigenform, the bounds are marginally easier to assemble, and we are left with computations that are shorter. Since this is generically the case if the genus of Q is of size two, we consider those forms first before considering the forms in a genus of size three.

2.2. Proof of Theorem 1.1: Genera of size two. Of the 14 quadratic forms whose regularity remains unproven, there are 11 that are in a genus of size two; see Table 2.1 for the list of these forms. As mentioned above, for each of these forms, the cuspidal part of the theta function is an eigenform, whose system of eigenvalues necessarily comes from a weight two newform. In fact, each of these newforms is associated to a rational elliptic curve, and we have indicated the Cremona label of each in Table 2.1. With this information and Chandee's bounds, it is now possible to put into action the approach described in the previous section.

Form	Disc.	Ell. Curve	Req. n	Time (s.)
$3x^2 + 6y^2 + 14z^2 + 4yz + 2xz + 2xy$	224	32a	$6.1 \cdot 10^6$	75
$x^2 + 5y^2 + 13z^2 + 2yz + xz + xy$	240	48a	$6.7 \cdot 10^6$	318
$x^2 + 6y^2 + 13z^2 + 3yz + xz$	297	99b	$2.7 \cdot 10^8$	3103
$2x^2 + 5y^2 + 11z^2 + 2yz + 2xz + xy$	405	27a	$1.1 \cdot 10^4$	0.2
$3x^2 + 5y^2 + 15z^2 + 3yz + 3xz + 3xy$	720	48a	$2.2 \cdot 10^7$	381
$x^2 + 10y^2 + 29z^2 + 5yz + xz$	1125	225b	$3.8 \cdot 10^8$	3508
$5x^2 + 8y^2 + 11z^2 - 4yz + xz + 2xy$	1620	27a	$8.5 \cdot 10^8$	23703
$2x^2 + 15y^2 + 32z^2 + 15yz + xz$	3375	225c	$8.3 \cdot 10^8$	6386
$5x^2 + 13y^2 + 33z^2 - 6yz + 3xz + xy$	8232	1176h	$7.2 \cdot 10^5$	47
$9x^2 + 11y^2 + 29z^2 - 4yz + 3xz + 6xy$	10125	225b	$9.4 \cdot 10^4$	3
$11x^2 + 15y^2 + 39z^2 - 3yz + 6xz + 3xy$	24696	1176h	$2.4 \cdot 10^6$	217

TABLE 2.1. The 11 forms in a genus of size two.

The form with the smallest discriminant is $Q := 3x^2 + 6y^2 + 14z^2 + 4yz + 2xz + 2xy$, with discriminant $224 = 2^5 \cdot 7$, and it is associated to the elliptic curve E with Cremona label 32a, given by the Weierstrass equation $E : y^2 = x^3 - x$. For each of the 32 square classes \mathbf{a} in $\mathbb{Q}_2^\times / (\mathbb{Q}_2^\times)^2 \times \mathbb{Q}_7^\times / (\mathbb{Q}_7^\times)^2$, we can find constants a , b , and d such that

$$r_Q(n) = a_E(n) + a_C(n) = ah(-bn) \pm dn^{1/4}L(1, E \otimes \chi_{-56n})^{1/2}$$

whenever $\bar{n} = \mathbf{a}$; if \mathbf{a} is not represented, then $a = d = 0$. As an example, if $\mathbf{a} = (3, 1)$, we find that $a = 1/4$, $b = 56$, and $d = 0.422\dots$. Using Dirichlet's class number formula, we have that

$$h(-56n) = \frac{1}{\pi} \sqrt{56n} L(1, \chi_{-56n}),$$

and so if n is not represented, we have that

$$\frac{L(1, E \otimes \chi_{-56n})^{1/2}}{L(1, \chi_{-56n})} \geq 1.409 \cdot n^{1/4}.$$

On the other hand, using Chandee's theorems, we find that

$$\frac{L(1, E \otimes \chi_{-56n})^{1/2}}{L(1, \chi_{-56n})} \leq 10.091 \cdot n^{0.124},$$

which implies that $n \leq 6.108 \cdot 10^6$. Similar computations for the other square classes yield either the same or smaller bounds on n , so it suffices to check that Q is regular for $n \leq 6.108 \cdot 10^6$. For convenience of computation, we note that elementary arguments imply that Q represents n if and only if $Q' := w^2 + 17y^2 + 10yz + 41z^2$ represents $3n$. We check the regularity of Q' up to $20 \cdot 10^6$ for integers divisible by 3, which takes 75 seconds.

The other forms of genus two are handled in exactly the same fashion. The required bounds on n are recorded in Table 2.1, along with the time required for the computation.

2.3. Proof of Theorem 1.1: Genera of size three. We now turn to the remaining three forms; see Table 2.2 for the list. For forms in a genus of size greater than two, we no longer expect the cuspidal part of the theta function to be an eigenform. Nonetheless, we are in the lucky situation that the cuspidal part of the first form, $Q := 5x^2 + 9y^2 + 15z^2 + 9yz + 3xz + 3xy$, happens to be an eigenform, so it can be dispatched as in Section 2.2; we have recorded the relevant data in Table 2.2. Moreover, while the cuspidal parts of the theta functions of the remaining two forms are not eigenforms under all Hecke operators $T(p^2)$ for $p \nmid N$, each is an eigenform under some $T(p^2)$. We exploit this fact to more easily compute the decomposition of $C(z)$ into eigenforms.

Form	Disc.	Ell. Curve(s)	Req. n	Time (s.)
$5x^2 + 9y^2 + 15z^2 + 9yz + 3xz + 3xy$	2160	48a	$6.7 \cdot 10^6$	320
$5x^2 + 9y^2 + 17z^2 + 6yz + 5xz + 3xy$	2592	32a, 288e	$2.4 \cdot 10^7$	1974
$5x^2 + 9y^2 + 27z^2 + 3xz + 3xy$	4536	56a, 504d	$7.0 \cdot 10^8$	30161

TABLE 2.2. The three forms in a genus of size three.

For the third form in Table 2.2, $Q := 5x^2 + 9y^2 + 27z^2 + 3xz + 3xy$, $C(z)$ is an eigenform under the Hecke operators $T(p^2)$ for $p = 5, 7$, and 13 , with eigenvalues $2, -1$, and 2 , respectively. There are only two newforms with these eigenvalues in the appropriate weight two spaces, and they are associated with the elliptic curves 56a and 504d. For convenience, we denote these two curves by E_1 and E_2 , respectively. Following the same approach as above, for each of the 128 classes $\mathbf{a} \in \mathbb{Q}_2^\times / (\mathbb{Q}_2^\times)^2 \times \mathbb{Q}_3^\times / (\mathbb{Q}_3^\times)^2 \times \mathbb{Q}_7^\times / (\mathbb{Q}_7^\times)^2$, we have, if $\bar{n} = \mathbf{a}$, that

$$r_Q(n) = ah(-bn) \pm d_1 n^{1/4} L(1, E_1 \otimes \chi_{-14n})^{1/2} \pm d_2 n^{1/4} L(1, E_2 \otimes \chi_{-14n})^{1/2},$$

where each of a, b, d_1 , and d_2 can be computed explicitly. We obtain for the squareclass $\mathbf{a} = (3, 2, 3)$, that

$$a = 1/4, \quad b = 56, \quad d_1 = 0.851\dots, \quad d_2 = 0.0801\dots,$$

which yields a bound of the form

$$\frac{d_1 \sqrt{L(1, E_1 \otimes \chi_{-14n})} + d_2 \sqrt{L(1, E_2 \otimes \chi_{-14n})}}{L(1, \chi_{-56n})} \geq 0.595 \cdot n^{1/4},$$

and, following Chandee, we have that

$$\frac{d_1 \sqrt{L(1, E_1 \otimes \chi_{-14n})} + d_2 \sqrt{L(1, E_2 \otimes \chi_{-14n})}}{L(1, \chi_{-56n})} \leq 7.743 \cdot n^{0.124}.$$

This yields a bound on n of $6.918 \cdot 10^8$. Similar computations reveal only smaller bounds.

For the second form in Table 2.2, $Q := 5x^2 + 9y^2 + 17z^2 + 6yz + 5xz + 3xy$, $C(z)$ is an eigenform for every $p \equiv 3 \pmod{4}$ with eigenvalue 0, indicating it is associated to weight

two CM cusp forms. There are eight such forms of possible level, each, again, associated to an elliptic curve. However, we find that of these eight systems of eigenvalues, only two play a role in $C(z)$; we have listed the Cremona data for each in Table 2.2. We follow the same technique as above, and have listed the relevant information in Table 2.2.

3. SIEGEL ZEROS: PROOF OF THEOREM 1.2

In this section, we consider the arithmetic consequences resulting from a large locally represented integer failing to be globally represented. The essential idea of the proof of Theorem 1.2 comes from equation (2.1), which states that

$$\theta_Q(z) = E(z) + C(z).$$

Similar to what we did in Section 2, by assuming the GRH for the family of modular L -functions arising from quadratic twists of newforms of conductor dividing $\Delta(Q)$, we are able to use Chandee's theorems [5] to obtain explicit upper bounds on the Fourier coefficients $a_C(n)$ of $C(z)$. Consequently, if n is locally represented, so that $a_E(n)$ is non-zero, and is not globally represented, so that $a_E(n) + a_C(n) = 0$, we obtain an explicit upper bound on $a_E(n)$, which would seemingly contradict the ineffective lower bound $a_E(n) \gg_\epsilon n^{1/2-\epsilon}$. Of course, the rectification of this comes from the fact that the implied constant depends upon a possible Siegel zero of some Dirichlet L -function. In particular, if there are no Siegel zeros $\sigma < 1$ satisfying

$$\sigma > 1 - \frac{c}{\log 3q},$$

then, following standard techniques (see Davenport [6, §21], for example) a lower bound of the form

$$(3.1) \quad h(d) \geq \alpha \cdot ce^{-8c} \frac{d^{1/2}}{\log^2 d}$$

can be obtained, where $\alpha = 1.288 \dots \cdot 10^{-4}$. Thus, if we are able to use the above ideas to contradict the bound (3.1), we will have produced a Siegel zero. This is obviously now our goal.

Following the techniques developed in Section 2, by applying Chandee's theorem, we obtain a bound of the form

$$a_C(n) \leq r \cdot n^s,$$

for some explicit constants r, s depending only on the class of n in $\prod_{p|\Delta} \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2$. In fact, by varying the parameter x in Chandee's bound [5, Equation (19)], we can obtain many different values of (r, s) , a fact which we will exploit for the purposes of optimization whenever dealing with a specific form – see the proof of Corollary 1.1 – but in general we only require x to be chosen so that $s < 1/2$. Provided that there is no contribution to $C(z)$ from a unary theta function, we are guaranteed to be able to make this choice (see equation (3.2) below), and if we restrict n to be squarefree and greater than the level of Q , we bypass this issue entirely. We could also require, if we write $n = n_0 n_1^2$ with n_0 squarefree, that n_0 is greater than the level, but we have chosen the statement we did for aesthetic purposes. At this point, Theorem 1.2 follows immediately. We now prove Corollary 1.1 to make this more explicit.

Proof of Corollary 1.1. We begin by noting that $Q = x^2 + y^2 + 10z^2$ has discriminant 40 and is associated to the elliptic curve $E : y^2 = x^3 + x^2 + 4x + 4$ with Cremona label 20a. For each square class $\mathbf{a} \in \mathbb{Q}_2^\times / (\mathbb{Q}_2^\times)^2 \times \mathbb{Q}_5^\times / (\mathbb{Q}_5^\times)^2$, and in particular each odd square class, we can find a, b , and d such that

$$r_Q(n) = ah(-bn) \pm d\sqrt{L(1, E \otimes \chi_{-10n})},$$

so that if n is not represented by Q , the bound

$$h(-bn) \leq \frac{d}{a}\sqrt{L(1, E \otimes \chi_{-10n})}$$

must hold. On the other hand, if there are no Siegel zeros for some $c < 1/8$, we also have the bound (3.1), so if the inequality

$$c \geq \frac{4ed}{a\alpha b^{1/2}} n^{-1/2} \log^2 n \sqrt{L(1, E \otimes \chi_{-10n})}$$

holds, we will have arrived at a contradiction. The bound for $L(1, E \otimes \chi_{-10n})$ obtained from Chandee's theorem is independent of the square class \mathbf{a} , and we compute that the constant is largest for the class $(1, 1)$, with $a = 2/3$, $b = 40$, and $d = 1.572\dots$. This yields that

$$c \geq 31480 \cdot n^{-1/2} \log^2 n \sqrt{L(1, E \otimes \chi_{-10n})}$$

will be problematic. As mentioned above, using Chandee's theorem, it is possible to bound the L -value by rn^s , where each of r and s depend upon a parameter x . In particular, we have that

$$(3.2) \quad s = \frac{1 + \lambda}{\log x}$$

and

$$(3.3) \quad \log r = \Re \left(\sum_{m \leq x} \frac{a(m)}{m^{\frac{1}{2} + \frac{\lambda}{\log x}} \log m} \frac{\log \frac{x}{m}}{\log x} \right) + \frac{2(\lambda^2 + \lambda)}{\log^2 x} + \frac{8e^{-\lambda}}{x^{\frac{1}{2}} \log^2 x} + \frac{1 + \lambda}{2 \log x} \log \left(\frac{800}{\pi^2} \right),$$

where $\frac{\log x}{2} \geq \lambda \geq \lambda_0$, $\lambda_0 = 0.4912\dots$ is the unique positive solution to $e^{-\lambda_0} = \lambda_0 + \lambda_0^2/2$, and the $a(m)$'s are the coefficients of the Dirichlet series

$$-\frac{L'}{L}(s, E \otimes \chi_{-10n}).$$

By taking $\lambda = \lambda_0$ and $x = 1000$, we obtain that $s = 0.215\dots$ and $r = 118.285\dots$. Thus, if $c \geq 342395 \cdot n^{-0.392} \log^2 n$, this yields a contradiction. However, we have assumed that $c \leq 1/8$, and we note that this bound is only below that if $n \geq 2.8 \cdot 10^{25}$. This is the stated result.

Moreover, the bounds on a, b , and d are all worst when $\mathbf{a} = (1, 1)$. Using the same values of λ and x , and that $\log n \leq e^{1/\epsilon} n^\epsilon$ for every $\epsilon > 0$, we find that if

$$n \geq 8.179 \cdot 10^{24} \cdot c^{-2.793},$$

then some Dirichlet L -function must have a Siegel zero. This establishes Theorem 1.2. \square

4. TATE-SHAFAREVICH GROUPS: PROOF OF THEOREM 1.3

We now turn our attention to the proof of Theorem 1.3. The starting point is again the inequality

$$h(-bn) \leq \frac{d}{a} \sqrt{L(1, E \otimes \chi_{-Dn})},$$

which must hold if n is in the locally represented square class \mathbf{a} but n is not globally represented. Here, we have assumed that Q is associated to the elliptic curve E . Instead of proceeding as we did in the proof of Theorem 1.2, however, we now assume GRH for Dirichlet L -functions, from which we obtain a lower bound for the central critical value of the modular L -function of the form

$$L(1, E \otimes \chi_{-Dn}) \geq \frac{a^2 \alpha^2 b}{64d^2 e^2} \frac{n}{\log^4(bn)} \gg \frac{n}{\log^4 n},$$

which should be compared to the Ramanujan bound $L(1, E \otimes \chi_{-d}) \ll_{E, \epsilon} d^\epsilon$. This inequality immediately guarantees that the analytic rank of $E \otimes \chi_{-Dn}$ is 0, which in turn yields that the arithmetic rank is 0 and the Tate-Shafarevich group $\text{III}(E \otimes \chi_{-Dn})$ is finite. However, if we want more control over the size of the Tate-Shafarevich group, we must assume the strong form of the Birch and Swinnerton-Dyer conjecture for rank 0 curves, which asserts, for any elliptic curve E' of rank 0, that

$$L(1, E') = \frac{\#\text{III}(E') \cdot \text{Tam}(E') \cdot \Omega(E')}{\#E'_{\text{tors}}(\mathbb{Q})^2},$$

where $\text{III}(E')$ denotes the Tate-Shafarevich group of E'/\mathbb{Q} , $\Omega(E')$ is the real period of E' , $\text{Tam}(E')$ is the Tamagawa number of E' , and $E'_{\text{tors}}(\mathbb{Q})$ denotes the rational torsion subgroup of E' . As E' varies over the family of quadratic twists of E , the torsion subgroup $E'_{\text{tors}}(\mathbb{Q})$ is bounded by Mazur's theorem. In fact, a stronger bound can be obtained – apart from possible 2-torsion, there are only finitely many twists with non-trivial torsion subgroup – but this is essentially irrelevant for our theorem. Moreover, the real period varies in a predictable manner; namely, we have that

$$\frac{\Omega(E \otimes \chi_{-d})}{\Omega(E \otimes \chi_{-4})} = d^{-1/2}.$$

While the Tamagawa numbers are harder to control, the general bound

$$\text{Tam}(E \otimes \chi_{-d}) \ll d^{1/2}$$

holds uniformly in E (see, e.g., [7]). The net effect of this is that, if n is locally represented but not globally, the inequality

$$\#\text{III}(E \otimes \chi_{-Dn}) \gg_E \frac{n}{\log^4 n}$$

must hold, where the implied constant can be made explicit. As mentioned in the introduction, this contradicts standard conjectures about the size of the Tate-Shafarevich group in the family of quadratic twists.

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DEPARTMENT OF MATHEMATICS, BUILDING 380, STANFORD UNIVERSITY, STANFORD, CA 94305
E-mail address: rjlo@stanford.edu