ETA-QUOTIENTS AND THETA FUNCTIONS

ROBERT J. LEMKE OLIVER

ABSTRACT. The Jacobi Triple Product Identity gives a closed form for many infinite product generating functions that arise naturally in combinatorics and number theory. Of particular interest is its application to Dedekind's eta-function $\eta(z)$, defined via an infinite product, giving it as a certain kind of infinite sum known as a theta function. Using the theory of modular forms, we classify all eta-quotients that are theta functions.

1. INTRODUCTION AND STATEMENT OF RESULTS

Jacobi's Triple Product Identity states that

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}z^2)(1+x^{2n-1}z^{-2}) = \sum_{n=-\infty}^{\infty} z^{2m}x^{m^2}, \tag{1.1}$$

which is surprising because it gives a striking closed form expression for an infinite product. Using (1.1), one can derive many elegant q-series identities. For example, one has Euler's identity

$$q\prod_{n=1}^{\infty}(1-q^{24n}) = \sum_{k=-\infty}^{\infty}(-1)^k q^{(6k+1)^2} \tag{1.2}$$

and Jacobi's identity

$$\prod_{n=1}^{\infty}\frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2} = \sum_{k=-\infty}^{\infty} q^{k^2}. \tag{1.3}$$

Both (1.2) and (1.3) can be viewed as identities involving Dedekind's eta-function $\eta(z)$, which is defined by

$$\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \tag{1.4}$$

where $q := e^{2\pi i z}$. It is well known that $\eta(z)$ is essentially a half-integral weight modular form, a fact which Dummit, Kisilevsky, and McKay [5] exploited to classify all the eta-products (functions of the form $\prod_{i=1}^{s}\eta(n_i z)^{t_i}$, where each n_i and each t_i is a positive integer) whose q-series have multiplicative coefficients. Martin [7] later obtained the complete list of integer weight eta-quotients (permitting the t_i to be negative) with multiplicative coefficients.

The right hand sides of both (1.2) and (1.3) also have an interpretation in terms of half-integral weight modular forms: they are examples of *theta functions*. Given a Dirichlet character ψ, the theta function $\theta_\psi(z)$ of ψ is given by

$$\theta_\psi(z) := \sum_{n}\psi(n)n^{\delta}q^{n^2}, \tag{1.5}$$

Key words and phrases. Modular forms; q-series; eta-quotients; theta functions.

where $\delta = 0$ or 1 according to whether ψ is even or odd. The summation over n in (1.5) is over the positive integers, unless ψ is the trivial character, in which case the summation is over all integers. With this language, (1.2) becomes

$$\eta(24z) = \theta_{\chi_{12}}(z),$$

where $\chi_{12}(n) = \left(\frac{12}{n}\right)$ and $\left(\frac{\cdot}{\cdot}\right)$ is the Jacobi symbol. This fact is subsumed into the theorem of Dummit, Kisilevsky, and McKay, as $\eta(24z)$ is an eta-product and any theta function necessarily has multiplicative coefficients. However, we note that (1.3) is equivalent to

$$\frac{\eta(2z)^5}{\eta(z)^2\eta(4z)^2} = \theta_1(z),$$

which is covered neither by the theorem of Dummit, Kisilevsky, and McKay (as the left-hand side is a quotient of eta-functions, not merely a product), nor is it covered by the theorem of Martin (as the modular forms involved are of half-integral weight). It is therefore natural to ask which eta-quotients are theta functions.

Theorem 1.1. *1. The following eta-quotients are the only ones which are theta functions for an even character:*

$$\begin{aligned}
\frac{\eta(2z)^5}{\eta(z)^2\eta(4z)^2} &= \sum_{n=-\infty}^{\infty} q^{n^2}, \\
\frac{\eta(8z)\eta(32z)}{\eta(16z)} &= \sum_{n=1}^{\infty} \left(\frac{2}{n}\right) q^{n^2}, \\
\frac{\eta(16z)^2}{\eta(8z)} &= \sum_{n=1}^{\infty} \left(\frac{n}{2}\right)^2 q^{n^2}, \\
\frac{\eta(6z)^2\eta(9z)\eta(36z)}{\eta(3z)\eta(12z)\eta(18z)} &= \sum_{n=1}^{\infty} \left(\frac{n}{3}\right)^2 q^{n^2}, \\
\eta(24z) &= \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{n^2}, \\
\frac{\eta(48z)^3}{\eta(24z)\eta(96z)} &= \sum_{n=1}^{\infty} \left(\frac{24}{n}\right) q^{n^2}, \\
\frac{\eta(48z)\eta(72z)^2}{\eta(24z)\eta(144z)} &= \sum_{n=1}^{\infty} \left(\frac{n}{6}\right)^2 q^{n^2}, \\
\frac{\eta(24z)\eta(96z)\eta(144z)^5}{\eta(48z)^2\eta(72z)^2\eta(288z)^2} &= \sum_{n=1}^{\infty} \left(\frac{18}{n}\right) q^{n^2}.
\end{aligned}$$

2. The following eta-quotients are the only ones which are theta functions for an odd character:

$$\eta(8z)^3 = \sum_{n=1}^{\infty} \left(\frac{-4}{n}\right) n q^{n^2},$$

$$\frac{\eta(16z)^9}{\eta(8z)^3\eta(32z)^3} = \sum_{n=1}^{\infty} \left(\frac{-2}{n}\right) nq^{n^2},$$
$$\frac{\eta(3z)^2\eta(12z)^2}{\eta(6z)} = \sum_{n=1}^{\infty} \left(\frac{n}{3}\right) nq^{n^2},$$
$$\frac{\eta(48z)^{13}}{\eta(24z)^5\eta(96z)^5} = \sum_{n=1}^{\infty} \left(\frac{-6}{n}\right) nq^{n^2},$$
$$\frac{\eta(24z)^5}{\eta(48z)^2} = \sum_{n=1}^{\infty} \left(\frac{n}{12}\right) nq^{n^2}.$$

In fact, we establish a broader classification theorem. Given a positive integer m, let Θ_m^0 denote the linear span of the set of all theta functions associated to an even character ψ whose modulus is m together with its 'twists' by $\chi_{2,0}$, $\chi_{3,0}$, and $\chi_{6,0}$, where $\chi_{r,0}$ denotes the principal character modulo r, and let Θ_m^1 denote the analogous space associated to odd characters of modulus m. Here, the twist of a theta function associated to ψ by another character χ is the theta function associated to $\psi\chi$. For convenience, let Θ_m denote the union of Θ_m^0 and Θ_m^1. We call an element of Θ_m monic if its q-expansion has the form $1 + O(q)$ or $q + O(q^4)$.

Theorem 1.2. *1. The only eta-quotients which are monic elements of Θ_m^0 for some m are those in Theorem 1.1 together with*

$$\frac{\eta(z)^2}{\eta(2z)} = \sum_{n=-\infty}^{\infty} \left(1 - 2\left(\frac{n}{2}\right)^2\right) q^{n^2} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2},$$
$$\frac{\eta(z)\eta(4z)\eta(6z)^2}{\eta(2z)\eta(3z)\eta(12z)} = \sum_{n=-\infty}^{\infty} \left(1 - \frac{3}{2}\left(\frac{n}{3}\right)^2\right) q^{n^2},$$
$$\frac{\eta(2z)^2\eta(3z)}{\eta(z)\eta(6z)} = \sum_{n=-\infty}^{\infty} \left(1 - 2\left(\frac{n}{2}\right)^2 - \frac{3}{2}\left(\frac{n}{3}\right)^2 + 3\left(\frac{n}{6}\right)^2\right) q^{n^2},$$
$$\frac{\eta(8z)^5}{\eta(4z)^2\eta(16z)^2} = \sum_{n=-\infty}^{\infty} \left(1 - \left(\frac{n}{2}\right)^2\right) q^{n^2},$$
$$\frac{\eta(9z)^2}{\eta(18z)} = \sum_{n=-\infty}^{\infty} \left(1 - 2\left(\frac{n}{2}\right) - \left(\frac{n}{3}\right)^2 + 2\left(\frac{n}{6}\right)^2\right) q^{n^2},$$
$$\frac{\eta(18z)^5}{\eta(9z)^2\eta(36z)^2} = \sum_{n=-\infty}^{\infty} \left(1 - \left(\frac{n}{3}\right)^2\right) q^{n^2},$$
$$\frac{\eta(4z)\eta(16z)\eta(24z)^2}{\eta(8z)\eta(12z)\eta(48z)} = \sum_{n=-\infty}^{\infty} \left(1 - \left(\frac{n}{2}\right)^2 - \frac{3}{2}\left(\frac{n}{3}\right)^2 + \frac{3}{2}\left(\frac{n}{6}\right)^2\right) q^{n^2},$$
$$\frac{\eta(72z)^5}{\eta(36z)^2\eta(144z)^2} = \sum_{n=-\infty}^{\infty} \left(1 - \left(\frac{n}{2}\right)^2 - \left(\frac{n}{3}\right)^2 + \left(\frac{n}{6}\right)^2\right) q^{n^2},$$

$$\frac{\eta(3z)\eta(18z)^2}{\eta(6z)\eta(9z)} = \sum_{n=1}^{\infty} \left(2\left(\frac{n}{6}\right)^2 - \left(\frac{n}{3}\right)^2\right) q^{n^2},$$

$$\frac{\eta(8z)^2\eta(48z)}{\eta(16z)\eta(24z)} = \sum_{n=1}^{\infty} \left(3\left(\frac{n}{6}\right)^2 - 2\left(\frac{n}{2}\right)^2\right) q^{n^2}.$$

2. The only eta-quotients which are monic elements of Θ_m^1 for some m are those in Theorem 1.1 together with

$$\frac{\eta(6z)^5}{\eta(3z)^2} = \sum_{n=1}^{\infty} \left(2\left(\frac{n}{12}\right) - \left(\frac{n}{3}\right)\right) nq^{n^2}.$$

Remark. A theorem of Mersmann [8] classifying holomorphic eta-quotients implies that there are essentially only finitely many eta-quotients which could be in any Θ_m, even allowing non-monic elements. Unfortunately, while Mersmann's result can be made effective, the computations necessary to prove either case of Theorem 1.2 in this way would be prohibitively large. Consequently, our proof proceeds along fundamentally different lines. We note that Mersmann's result is slightly misquoted in [3] - the theorem credited to Mersmann on Page 30 of [3] is stronger than what he proves in his thesis.

Our proof proceeds as follows. Instead of using the method employed by Mersmann [8] - essentially a careful study of the order of vanishing of eta-quotients - we make use of the combinatorial properties of eta-quotients and the constraints on the q-series of theta functions. Combined with the theory of modular forms, in particular the Fricke involution $W_{k,M}$, asymptotic formulae, Eisenstein series, and Shimura's correspondence, the classification in Theorems 1.1 and 1.2 reduces to a case by case analysis. In this analysis, we make great use of the simple observation that if $a > b$, then $(1+q^a)(1+q^b) = 1 + q^b + O(q^a)$. In this regard, we also need the solution to a classical Diophantine problem.

Acknowledgements

The author would like to thank Ken Ono, who suggested this problem to him, Marie Jameson, Jeremy Rouse, and Karl Mahlburg, for useful discussions, and the anonymous referees, who suggested considerable improvements.

2. Preliminary Facts

We begin by recalling some basic facts about modular forms. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a weakly holomorphic modular form of weight $k \in \frac{1}{2}\mathbb{Z}$ for the subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ if $f(\gamma z) = \epsilon_\gamma (cz+d)^k f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, acting in the usual way by fractional linear transformation, where ϵ_γ is a suitable fourth root of unity. Moreover, we require for each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ that $(f \mid_k \gamma)(z) := (cz+d)^{-k} f(\gamma z)$ is represented by a Fourier series of the form

$$(f \mid_k \gamma)(z) = \sum_{n \geq n_0} a_\gamma(n) q_N^n,$$

where $q_N := e^{2\pi i/N}$. In fact, there are only finitely many such series required, one for each "cusp" of $\Gamma \setminus \mathbb{H}$, that is, an element $\rho \in \Gamma \setminus \mathbb{Q}$. In this case, we let $q_\rho := q_N$. The space of all weakly holomorphic modular forms of weight k on Γ is denoted by $M_k^!(\Gamma)$; its subspace consisting of all forms which are holomorphic (resp. vanishing) at the cusps is denoted by $M_k(\Gamma)$ (resp. $S_k(\Gamma)$) (these are the spaces of modular forms and cusp forms, respectively). For the subgroups we are concerned with, namely the congruence subgroups of level N,

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\},$$

weakly holomorphic modular forms are fixed under the substitution $z \mapsto z+1$, and so they have a Fourier series at infinity with respect to the variable $q := e^{2\pi i z}$. Although we will briefly need the Fourier expansions at other cusps, it is this Fourier series (also called a q-expansion) that is of the most interest to us.

If $f(z) \in M_k^!(\Gamma_1(N))$, then $f(z)$ is said to be modular of level N. If $f(z)$ is holomorphic, then necessarily $k \geq 0$ and the space $M_k(\Gamma_1(N))$ decomposes naturally into two pieces: the previously mentioned cusp space $S_k(\Gamma_1(N))$ and the so-called Eisenstein space $\mathcal{E}_k(\Gamma_1(N))$. If $k \in \mathbb{Z}$, exploiting this decomposition, the size of the Fourier coefficients $a_f(n)$ of $f(z)$ is well-understood. In particular, letting $a_f(n) = a_{\text{cusp}}(n) + a_{\text{Eis}}(n)$, then we have that both

$$a_{\text{Eis}}(n) \ll_{f,\epsilon} n^{k-1+\epsilon},$$

due to the explicit nature of the coefficients (see (2.5) below), and

$$a_{\text{cusp}}(n) \ll_{f,\epsilon} n^{(k-1)/2+\epsilon},$$

which is the celebrated bound of Deligne. If $k \in 1/2 + \mathbb{Z}$, then the coefficients of both the Eisenstein series and the cuspidal part are not understood nearly as well, as both frequently encode values of L-functions. Nevertheless, polynomial bounds are known for each. In particular, we have the "trivial" bound that

$$a_{\text{cusp}}(n) \ll_f n^{k/2},$$

valid for all $k \geq 1/2$ and

$$a_{\text{Eis}}(n) \ll_{f,\epsilon} n^{k-1+\epsilon}$$

for $k \geq 3/2$, and $a_{\text{Eis}}(n) \ll n^{\epsilon}$ if $k = 1/2$. Although stronger bounds are known (most recently, due to Blomer and Harcos [1]), it is only the fact that each is polynomially bounded that will be relevant to us. This is because if $f(z)$ is weakly holomorphic, but not holomorphic, then the coefficients of $f(z)$ are of a fundamentally different size. Namely, for n in certain arithmetic progressions depending on the level, they satsify

$$\log |a_f(n)| \gg n^{1/2}.$$

This is due, in various settings, to Rademacher and Zuckerman ([11], [12], [14], [15]), and, more recently, to Bringmann and Ono [2]. We will find this vast difference in size useful later on.

We recall that, given a Dirichlet character ψ of modulus r, the function $\theta_\psi(z)$ is defined by $\theta_\psi(z) := \sum_n \psi(n) n^{\delta} q^{n^2}$, where $\delta = 0$ or 1 according to whether ψ is even or odd. In both cases, the summation over n is assumed to be over the positive integers unless $r = 1$, in which case the sum is over all integers. It is classical that $\theta_\psi(z)$ is a modular form of weight 1/2 if ψ is even and of weight 3/2 if ψ is odd. Each $\theta_\psi(z)$ is of level $4r^2$ and, moreover, if $r \neq 1$, then it is a cusp form. Regardless of the parity of ψ, we refer to $\theta_\psi(z)$ as a theta function of modulus r.

The twist of a theta function $\theta_\psi(z)$ by a character χ is the theta function associated to $\psi\chi$. Given a positive integer m, we let Θ_m^0 denote the linear span of the set of all weight 1/2 theta functions whose moduli are m and their twists by $\chi_{2,0}$, $\chi_{3,0}$, and $\chi_{6,0}$, and we let Θ_m^1 denote the analogous space for weight $3/2$ theta functions. Let Θ_m be the union of Θ_m^0 and Θ_m^1.

Dedekind's eta-function $\eta(z)$ is defined by

$$\eta(z) := q^{1/24} \prod_n (1 - q^n).$$

It is almost a modular form of weight $1/2$ on $SL_2(\mathbb{Z})$, in the sense that

$$\eta\left(\frac{-1}{z}\right) = (-iz)^{1/2}\eta(z), \tag{2.1}$$

but it fails to transform suitably under $z \mapsto z+1$. However, since $\eta(z)$ is non-vanishing away from the cusps, a function of the form

$$f(z) = \prod_{d|N} \eta(dz)^{r_d} \tag{2.2}$$

will be a weakly holomorphic modular form on $\Gamma_1(24N)$ if $\sum_{d|N} dr_d \equiv 0 \pmod{24}$. This level may not be sharp, in the sense that $f(z)$ may be a weakly holomorphic modular form on $\Gamma_1(M)$ for some proper divisor M of $24N$, but what is important for our purposes is that the only primes dividing the level of $f(z)$ are those dividing N together with 2 and 3. We call a function of the form (2.2) satisfying this condition an eta-quotient. The order of vanishing of $f(z)$ at the cusp $\rho := \frac{\alpha}{\delta}$ is given by [10, Theorem 1.65]

$$\mathrm{ord}_{z=\rho} f(z) = \frac{N}{24} \sum_{d|N} \frac{(d,\delta)^2 r_d}{(\delta, \frac{N}{\delta}) d\delta}. \tag{2.3}$$

Lemma 2.1. *Suppose that $f(z) = \prod_{d|N} \eta(dz)^{r_d}$ is an element of Θ_m. Set $a := 1 + \max(1, \nu_2(m))$ and $b := \max(1, \nu_3(m))$, where $\nu_p(\cdot)$ is the standard p-adic valuation, and let m_0 be the maximal divisor of m coprime to 6. Then $r_d = 0$ for $d \nmid 2^{2a}3^{2b}m_0^2$.*

Proof. Given a theta function $\theta(z)$ of modulus r and weight k, it is well known that $\theta(z) \mid W_{k,4r^2} := (-2rz)^{-k}\theta\left(\frac{-1}{4r^2z}\right)$ is again a modular form of weight k whose Fourier series has integral exponents, where $W_{k,4r^2}$ is the usual Fricke involution (see [10, Proposition 3.8]). This property also holds for $\theta(z)|W_{k,4r^2t}$ for any t, and so we see that the operator $W_{k,2^{2a}3^{2b}m_0^2}$ sends Θ_m to the union of two spaces of modular forms whose Fourier series have integral exponents. If $f(z)$ is in Θ_m and has weight k, therefore, we must have that $f(z)|W_{k,2^{2a}3^{2b}m_0^2}$ is a modular form of weight k with only integral exponents.

On the other hand, we compute using (2.1) that

$$\begin{aligned} f(z)|W_{k,2^{2a}3^{2b}m_0^2} &= (-2^a3^bm_0z)^{-k} \prod_{d|N} \eta\left(\frac{-d}{2^{2a}3^{2b}m_0^2 z}\right)^{r_d} \\ &= C\prod_{d|N} \eta\left(\frac{2^{2a}3^{2b}m_0^2 z}{d}\right)^{r_d} =: C\tilde{f}(z) \end{aligned}$$

for some constant C. But the only way for $\tilde{f}(z)$ to have a Fourier series with integral exponents is if for each $d \nmid 2^{2a}3^{2b}m_0^2$ we have that $r_d = 0$. $\square$

The above lemma limits the eta-quotients which are in Θ_m for a fixed m, but we still need a way to control the possible values of m. The following proposition permits us to do that. First, though, we fix notation. Given a weakly holomorphic modular form $f(z) = \sum_{n\gg-\infty} a(n)q^n$ of level N, the U_p operator for a prime p is defined by

$$f(z)|U_p := \sum_{n\gg-\infty} a(pn)q^n.$$

It is well known that $f(z)|U_p$ is again a weakly holomorphic modular form of level N if $p \mid N$ and level pN if $p \nmid N$.

Proposition 2.2. *If $f \in M_k^!(\Gamma_1(N))$ and $p \nmid N$ is prime, then $f(z)|U_p = 0$ if and only if $f(z) = 0$.*

Before proving Proposition 2.2, we deduce its application to the problem at hand.

Corollary 2.3. *If $f(z) = \prod_{d|N} \eta(dz)^{r_d}$ is in Θ_m for some m, then the only primes dividing m are 2 and 3.*

Proof. We first show that any eta-quotient $f(z)$ is not annihilated by the U_p operator for any $p \geq 5$. Fix such a prime and write $f(z) = f_1(z)f_2(pz)$, with

$$\begin{aligned} f_1(z) &:= \prod_{d|N, p\nmid d} \eta(dz)^{r_d}, \text{ and} \\ f_2(z) &:= \prod_{d|N, p|d} \eta\left(\frac{dz}{p}\right)^{r_d}, \end{aligned}$$

where an empty product has value 1. We now have that $f(24z)|U_p = f_1(24z)|U_p \cdot f_2(24z)$. Since $f_1(24z)$ is a weakly holomorphic modular form of level indivisible by p, we see by Proposition 2.2 that $f_1(24z)|U_p$ is non-zero, and since $f_2(24z) \neq 0$, we also have that $f(24z)|U_p$ is non-zero. Since $p \nmid 24$ by assumption, it follows that that $f(z)|U_p \neq 0$.

On the other hand, functions in Θ_m have the property that their coefficients are supported on exponents which are coprime to m. Hence, for any prime divisor p of m, we must have that U_p annihilates Θ_m. Thus, if $f(z)$ is in Θ_m, the only way it can have this property is if the only primes dividing m are 2 and 3. □

The proof of Proposition 2.2 relies upon a lemma on sums of almost-everywhere multiplicative functions, which we define to be functions satisfying $f(mn) = f(m)f(n)$ for any coprime m and n, neither of which is divisible by any of a finite set of primes called the *bad primes*. As an example, if $f(n)$ is a multiplicative function such that $f(t) \neq 0$ for some t, then $f(tn)/f(t)$ is not generically multiplicative. It is, however, almost-everywhere multiplicative away from the primes dividing t.

Lemma 2.4. *Suppose that $f_1, \cdots, f_s$ are almost-everywhere multiplicative functions which are each non-zero for an infinite set of primes. Moreover, assume that, for each $i \neq j$, $f_i(p) \neq f_j(p)$ for an infinite number of primes p. If $c_1f_1(n) + \ldots + c_sf_s(n) = 0$ for all n indivisible by every bad prime, then each $c_i = 0$.*

Proof. We proceed by induction, noting that the result is obviously true if $s = 1$.

If $s \geq 2$, we may assume by way of contradiction that each $c_i \neq 0$, so we have that

$$f_s(n) = -\sum_{i=1}^{s-1} \frac{c_i}{c_s} f_i(n) \tag{2.4}$$

for every n not divisible by any bad prime. Let m and n be coprime integers not divisible by any bad prime. We then have that both

$$f_s(mn) = -\sum_{i=1}^{s-1} \frac{c_i}{c_s} f_i(m) f_i(n)$$

and

$$\begin{aligned} f_s(mn) &= \left(-\sum_{i=1}^{s-1} \frac{c_i}{c_s} f_i(m)\right)\left(-\sum_{i=1}^{s-1} \frac{c_i}{c_s} f_i(n)\right) \\ &= \sum_{i=1}^{s-1} \left(\frac{c_i}{c_s} \sum_{j=1}^{s-1} \frac{c_j}{c_s} f_j(m)\right) f_i(n). \end{aligned}$$

Equating these two expressions for $f_s(mn)$, we obtain that

$$\sum_{i=1}^{s-1} \left(\frac{c_i}{c_s} f_i(m) + \frac{c_i}{c_s} \sum_{j=1}^{s-1} \frac{c_j}{c_s} f_j(m)\right) f_i(n) = 0,$$

which, by our induction hypothesis, can only happen if for each i and m, we have that

$$\frac{c_i}{c_s} \left(f_i(m) + \sum_{j=1}^{s-1} \frac{c_j}{c_s} f_j(m)\right) = 0.$$

Since $c_i \neq 0$ for each i, we then get a linear combination of $f_j(m)$ equaling 0, and again using the induction hypothesis, we find that $c_i = -c_s$ and all other $c_j = 0$. Since we assumed that each $c_j \neq 0$, this can only happen if $s = 2$, and in that case, (2.4) yields that $f_1(n) = f_2(n)$ for all n away from the set of bad primes. But since these functions were assumed to be distinct, this cannot happen. □

Proof of Proposition 2.2. If the Fourier expansion of $f(z)$ at the cusp ρ is given by

$$f(z) = \sum_{n \gg -\infty} a_\rho(n) q_\rho^{n+\kappa_\rho},$$

then the principal part of $f(z)$ at ρ is

$$f_\rho^-(z) = \sum_{n+\kappa_\rho < 0} a_\rho(n) q_\rho^{n+\kappa_\rho}.$$

Following either the classical work of Rademacher and Zuckerman ([11], [12], [14], [15]) or the recent work of Bringmann and Ono [2], we can write $f(z) = f^-(z) + f_{\text{hol}}(z)$, where $f^-(z)$ is a linear combination of so-called Maass-Poincaré series which matches the principal part of $f(z)$ at each cusp and $f_{\text{hol}}(z)$ is a holomorphic modular form. Since the coefficients of the Maass-Poincaré series grow superpolynomially along certain arithmetic progressions modulo N and the coefficients of $f_{\text{hol}}(z)$ are polynomially bounded (see the above discussion), in order for $f(z)|U_p$ to be 0, we must have that $f^-(z) = 0$ and $f(z) = f_{\text{hol}}(z)$. In the case that $k < 0$, we are now done, as there are no holomorphic modular forms of negative weight. If

$k=0$, the only holomorphic modular forms are constant and are preserved under U_p. Hence, in this case too, we must have that $f(z)=0$.

If $k=1/2$, a deep theorem of Serre and Stark [13] states that $f(z)$ must be a linear combination of weight 1/2 theta functions $\theta_\chi(z)$ of level dividing N and their dilates $\theta_\chi(tz)$, with $t\cdot\mathrm{cond}(\chi)\mid N$. Thus, if $(n,N)=1$, we have that

$$0=a_f(p^2n^2)=\sum_\chi c_\chi\chi(p)\chi(n),$$

where the sum runs over the characters of conductor dividing N, and each c_χ is a constant. But by the linear independence of characters, we must have that each $c_\chi\chi(p)=0$, whence $c_\chi=0$ since $(p,\mathrm{cond}(\chi))=1$. Considering iteratively $a_f(tp^2n^2)$ in the same way, the result follows if $k=1/2$.

We may now suppose that $k\geq 1$. In the case that k is an integer, following [4], a basis for the Eisenstein space of $M_k(\Gamma_1(N))$ is given by

$$\{E_k^{\varepsilon,\psi,t}(z):(\varepsilon,\psi,t)\in A_{k,N}\},$$

where we define $E_k^{\varepsilon,\psi,t}(z)$ using the series

$$E_k^{\varepsilon,\psi}(z)=c_{k,\varepsilon,\psi}+\sum_{n=1}^{\infty}\left(\sum_{d|n}\varepsilon(n/d)\psi(d)d^{k-1}\right)q^n \tag{2.5}$$

by $E_k^{\varepsilon,\psi,t}(z)=E_k^{\varepsilon,\psi}(tz)$ for $k\neq 2$ or $(\varepsilon,\psi)\neq(1,1)$, and $E_2^{1,1,t}(z)=E_2^{1,1}(z)-tE_2^{1,1}(tz)$ for $t\neq 1$. In the above, $c_{k,\varepsilon,\psi}$ is a constant and $A_{k,N}$ is the set of triples (ε,ψ,t) where ε and ψ are primitive Dirichlet characters of conductor u and v, respectively, with $(\varepsilon\psi)(-1)=(-1)^k$, and t is a positive integer such that $tuv\mid N$. In the case $k=2$ we exclude the triple $(1,1,1)$, and in the case $k=1$, we require the first two elements of a triple to be unordered. We note that the Fourier coefficients of the Eisenstein series $E_k^{\varepsilon,\psi}(z)$ are multiplicative; we denote these coefficients by $\sigma_{k-1}^{\varepsilon,\psi}(n)$.

For the cusp space $S_k(\Gamma_1(N))$, we may choose a basis of Hecke eigenforms, so that any holomorphic modular form is a linear combination of forms $g_1(z),\cdots,g_s(z)$, each with Fourier coefficients $a_1(n),\cdots,a_s(n)$ that are "essentially" multiplicative: in the case that $a_i(n)$ arises from a newform (that is, a form not coming from some level $M\mid N$), it is legitimately multiplicative, but if $a_i(n)$ arises from a non-newform, then it is of the form $a_i(tn)=a_j(n)$ for some $t\mid N$ and the coefficients $a_j(n)$ of some newform (the role of t will be handled easily in the proof below). In particular, if $f(z)=\sum a(n)q^n$, we may, for $(n,N)=1$, write

$$a(n)=c_1f_1(n)+\cdots c_rf_r(n)$$

for some constants $c_1,\cdots,c_r$ and multiplicative $f_i(n)$, coming either from an Eisenstein series or an eigenform. Since $f(z)|U_p=0$, we must have that $a(pn)=0$ for all n. In particular, for $(n,pN)=1$, we must have that

$$0=c_1f_1(pn)+\cdots+c_rf_r(pn)=c_1f_1(p)f_1(n)+\cdots+c_rf_s(p)f_r(n),$$

and we may omit any $f_i(n)$ arising from an Eisenstein series $E_k^{\varepsilon,\psi}(tz)$ with $t>1$ or from a non-newform (since $(n,N)=1$ and $t\mid N$, the omitted coefficients are 0 and will have no effect on $a(pn)$). By Lemma 2.4, we must have that each $c_if_i(p)=0$. Now, $f_i(p)$ may be zero for some i, but in that case we necessarily have that $f_i(p^2)\neq 0$ (this follows from the Euler product expansion in the case of a cusp form and by a direct computation in the case of Eisenstein series, which can only have this property if $k = 1$). Thus, by also considering $a(p^2n)$, we see that each c_i not arising from an $E_k^{\varepsilon,\psi}(tz)$ or a non-newform must be 0. Iteratively letting t be the smallest divisor of N not yet considered and repeating the argument above for $a(tpn)$ and $a(tp^2n)$, we see that all $c_i = 0$, whence $f(z) = 0$ identically.

In the case that the weight is half-integral, we may still choose as a basis of the cusp space a sequence of Hecke eigenforms, but we no longer know that there is a basis of the Eisenstein space with multiplicative Fourier coefficients. We proceed, therefore, to show that $f(z)$ is a cusp form. The argument in the integer weight case then applies, showing that $f(z) = 0$.

Consider the image of $f(z)$ under the Shimura map $S_{\lambda,\tau} : M_{\lambda+\frac{1}{2}}(\Gamma_1(N)) \to M_{2\lambda}(\Gamma_1(N))$, where $\lambda := k - 1/2$ and τ is any squarefree positive integer (often, this map is only defined for the cusp-space; see, for example, work of Jagathesan and Manickam [6] on extensions of this). If $f(z) = \sum a(n)q^n$, the non-constant terms of $S_{\lambda,\tau}(f(z)) = \sum b(n)q^n =: F(z)$ are given by the Dirichlet series formula [10, Theorem 3.14]

$$\sum_{n=1}^{\infty} b(n)n^{-s} = L(s+1-\lambda, \chi_\tau) \sum_{n=1}^{\infty} a(\tau n^2)n^{-s},$$

where χ_τ is a Dirichlet character. Hence, we have that

$$b(n) = \sum_{d|n} d^{\lambda-1}\chi_\tau(d)a\left(\frac{\tau n^2}{d^2}\right).$$

In particular if $(p, n) = 1$, then

$$\begin{aligned} b(p^m n) &= \sum_{d|p^m n} d^{\lambda-1}\chi_\tau(d)a\left(\frac{\tau p^{2m} n^2}{d^2}\right) \\ &= \sum_{d|n} (p^m d)^{\lambda-1}\chi_\tau(p^m d)a\left(\frac{\tau n^2}{d^2}\right) \\ &= p^{m(\lambda-1)}\chi_\tau(p^m)b(n), \end{aligned}$$

since $a(pr) = 0$ for any r. Let $F_0(z)$ denote the projection of $F(z)$ into the Eisenstein space of $M_{2\lambda}(\Gamma_1(N))$. We can express $F_0(z)$ as

$$F_0(z) = \sum_{(\varepsilon,\psi,t)} a_{\varepsilon,\psi,t} E_{2\lambda}^{\varepsilon,\psi}(tz), \tag{2.6}$$

where we are summing over all triples (ε, ψ, t) such that ε and ψ are primitive characters of conductor dividing N and t is any divisor of N (thus, we include triples (ε, ψ, t) which do not arise in $A_{2\lambda,N}$). We require that $a_{\varepsilon,\psi,t} = 0$ when $(\varepsilon, \psi, t) \notin A_{2\lambda,N}$, unless $\lambda = 1$, where we let $a_{1,1,1}$ absorb the coefficients of $E_2^{1,1}(z)$ arising from the terms $E_2^{1,1,t}(z)$ in the basis expansion of the Eisenstein space of $M_2(\Gamma_1(N))$. We then have that, if $(n, pN) = 1$,

$$b(p^m n) = \sum_{\varepsilon,\psi} a_{\varepsilon,\psi,1}\sigma_{2\lambda-1}^{\varepsilon,\psi}(p^m n) + O\left((p^m n)^{\lambda-1/2+\epsilon}\right)$$

by Deligne's bound. Similarly, we also have for such n that

$$b(n) = \sum_{\varepsilon,\psi} a_{\varepsilon,\psi,1}\sigma_{2\lambda-1}^{\varepsilon,\psi}(n) + O\left(n^{\lambda-1/2+\epsilon}\right).$$

Since $b(p^m n) = p^{m(\lambda-1)}\chi_\tau(p^m)b(n)$, we must have that

$$\sum_{\varepsilon,\psi} \tilde{a}_{\varepsilon,\psi}\sigma_{2\lambda-1}^{\varepsilon,\psi}(n) = O\left((p^m n)^{\lambda-1/2+\epsilon}\right), \tag{2.7}$$

where

$$\tilde{a}_{\varepsilon,\psi} = a_{\varepsilon,\psi,1}\cdot(\sigma_{2\lambda-1}^{\varepsilon,\psi}(p^m) - p^{m(\lambda-1)}\chi_\tau(p^m)).$$

Let $n = \ell_1\ell_2$, where ℓ_1 and ℓ_2 are large primes such that $\ell_2 \asymp \ell_1^{1/2}$. Then (2.5) implies that

$$\begin{aligned}\sigma_{2\lambda-1}^{\varepsilon,\psi}(\ell_1\ell_2) &= (\psi(\ell_1)\ell_1^{2\lambda-1} + \varepsilon(\ell_1))\cdot(\psi(\ell_2)\ell_2^{2\lambda-1} + \varepsilon(\ell_2)) \\ &= \psi(\ell_1)\psi(\ell_2)(\ell_1\ell_2)^{2\lambda-1} + \psi(\ell_1)\varepsilon(\ell_2)\ell_1^{2\lambda-1} + O\left(\ell_1^{\lambda-1/2}\right).\end{aligned}$$

Using this in (2.7) and dividing by $(\ell_1\ell_2)^{2\lambda-1}$, we see that

$$\sum_{\varepsilon,\psi} \tilde{a}_{\varepsilon,\psi}\psi(\ell_1)\psi(\ell_2) = O\left(\ell_1^{-\lambda+1/2} + p^{m(\lambda-1/2)+\epsilon}\ell_1^{-\lambda+1/2+\epsilon}\right),$$

and letting ℓ_1 and ℓ_2 tend to infinity along fixed arithmetic progressions modulo N, we see that, in fact,

$$\sum_{\varepsilon,\psi} \tilde{a}_{\varepsilon,\psi}\psi(\ell_1)\psi(\ell_2) = 0.$$

Hence, (2.7) and the expansion of $\sigma_{2\lambda-1}(\ell_1\ell_2)$ now yield that

$$\sum_{\varepsilon,\psi} \tilde{a}_{\varepsilon,\psi}\psi(\ell_1)\varepsilon(\ell_2) = O\left(\ell_1^{-\lambda+1/2} + p^{m(\lambda-1)+\epsilon}\ell_1^{-\frac{1}{2}(\lambda-1/2)+\epsilon}\right),$$

and again letting ℓ_1 and ℓ_2 tend to infinity along fixed arithmetic progressions, we see that

$$\sum_{\varepsilon,\psi} \tilde{a}_{\varepsilon,\psi}\psi(\ell_1)\varepsilon(\ell_2) = 0. \tag{2.8}$$

Since ℓ_1 and ℓ_2 were chosen to be in arbitrary arithmetic progressions modulo N, this can be viewed as an equation in terms of the matrix $K_N \otimes K_N$, where K_N is the $\phi(N)\times\phi(N)$ matrix whose components are $\chi(a)$ as a runs over elements of $(\mathbb{Z}/N\mathbb{Z})^\times$ and χ runs over its characters. Since K_N is invertible (it is the tensor product of Vandermonde matrices arising from the cyclic factors of $(\mathbb{Z}/N\mathbb{Z})^\times$), $K_N \otimes K_N$ is as well. Hence, the only way for (2.8) to hold is if each $\tilde{a}_{\varepsilon,\psi} = 0$. Recall that

$$\tilde{a}_{\varepsilon,\psi} = a_{\varepsilon,\psi,1}\cdot(\sigma_{2\lambda-1}^{\varepsilon,\psi}(p^m) - p^{m(\lambda-1)}\chi_\tau(p^m)),$$

and since $\sigma_{2\lambda-1}^{\varepsilon,\psi}(p^m) \asymp p^{m(2\lambda-1)}$, by considering large enough m, we conclude that each $a_{\varepsilon,\psi,1} = 0$. By iteratively letting t be the smallest divisor of N not yet considered and looking at $b(tp^m n)$, the above argument shows that each $a_{\varepsilon,\psi,t} = 0$. Consequently, we must have that $F_0(z) = 0$ and $F(z)$ is a cusp form. But since this is true independent of the choice of τ in the map $S_{\lambda,\tau}$, we also have that $f(z)$ is itself a cusp form. We now proceed as in the integer-weight case. Strictly speaking, the coefficients of half-integer weight eigenforms are only almost-everywhere multiplicative in square classes (that is, for t squarefree, $a(tn^2)/a(t)$ is multiplicative in n, provided that $(n, N) = 1$), but by considering each square class separately, the result follows. □

Before we can prove Theorem 1.2, we need one further lemma.

Lemma 2.5. *The only $a = 2^i 3^j$ which are one less than a square are $a = 3, 8, 24, 48,$ and 288.*

Proof. Suppose $a = n^2 - 1 = (n+1)(n-1)$. If a is odd, then we must have that both $n-1$ and $n+1$ are powers of 3, and so $a = 3$. If a is not divisible by 3, then both $n-1$ and $n+1$ must be powers of 2, and so $a = 8$. Lastly, suppose that a is divisible by both 2 and 3. Since both $n-1$ and $n+1$ are then required to be even, the pair $\left(\frac{n-1}{2}, \frac{n+1}{2}\right)$ must be either $(2^{i-2}, 3^j)$ or $(3^j, 2^{i-2})$. It is a classical result due to Levi ben Gerson that the only powers of 2 and 3 which differ by one are $(2,3)$, $(3,4)$, and $(8,9)$. These lead to $a = 24$, 48, and 288, respectively. $\square$

Remark. Of course, by Mihăilescu's resolution of Catalan's conjecture [9] it is known that 8 and 9 are the only consecutive perfect powers.

3. Proof of Theorem 1.2

We begin by considering a few concrete cases of Theorem 1.1. Although we could prove Theorem 1.2 without doing so, this allows us to illustrate the constructive approach we shall take.

Lemma 2.1 and Corollary 2.3 together tell us that if $f(z) = \prod_{d|N} \eta(dz)^{r_d}$ is an eta-quotient which is also a theta function $\theta_\psi(z)$ of modulus r, then r must be divisible only by the primes 2 and 3, and we may take $N = 4r^2$. Since the coefficients of any eta-quotient are real, we must also have that ψ is a quadratic character. The key observation which leads to Theorem 1.1 is that if we know the modulus of a quadratic theta function, then we know the first few terms in its Fourier series, at least up to a sign.

First, we consider whether $\theta_1(z) = 1 + 2\sum_{n=1}^\infty q^{n^2}$ is an eta-quotient. Let $\eta_0(z) := \prod_{n=1}^\infty (1 - q^n)$, so that $f(z) = q^{a_f} \prod_{d|N} \eta_0(dz)^{r_d}$, where $a_f = \sum_{d|N} \frac{dr_d}{24}$. In order for $f(z)$ to be equal to $\theta_1(z)$, we must have that $a_f = 0$, as $\eta_0(dz) = 1 + O(q^d)$. In fact, we know more:

$$\eta_0(dz)^{r_d} = 1 - r_d q^d + O(q^{2d}). \tag{3.1}$$

Consequently, in order for the Fourier expansion of $f(z)$ to match that of

$$\theta_1(z) = 1 + 2q + 2q^4 + 2q^9 + O(q^{16}),$$

we must have that $r_1 = -2$ or, equivalently, that -2 is the exact power of $\eta(z)$ dividing $f(z)$ (we say that $\eta(z)^{-2}$ divides $f(z)$). The Fourier series of $\eta_0(z)^{-2}$ is given by

$$\eta_0(z)^{-2} = 1 + 2q + 5q^2 + O(q^3),$$

and since there is no q^2 term in the Fourier expansion of $\theta_1(z)$, we see that $\eta(2z)^5$ must also divide $f(z)$ in order to cancel the $5q^2$ term in the Fourier series for $\eta_0(z)^{-2}$. This leads us to consider

$$\frac{\eta_0(2z)^5}{\eta_0(z)^2} = 1 + 2q - 4q^5 + O(q^6).$$

Since we now need to add $2q^4$ to this Fourier expansion, we see that $\eta(4z)^{-2}$ must also divide $f(z)$. We compute that

$$\frac{\eta_0(2z)^5}{\eta_0(z)^2 \eta_0(4z)^2} = 1 + 2q + 2q^4 + 2q^9 + O(q^{16}),$$

which matches perfectly the Fourier expansion of $\theta_1(z)$! Since for $f(z) = \frac{\eta(2z)^5}{\eta(z)^2\eta(4z)^2}$, we have that $a_f = 0$, $f(z)$ is a candidate for an eta-quotient representation of $\theta_1(z)$. One easily verifies via (2.3) that $f(z)$ is holomorphic, and then the Sturm bound [10, Theorem 2.58] implies that $f(z) = \theta_1(z)$.

We now suppose that ψ is a character whose modulus is a positive power of 2 and that the weight of $\theta_\psi(z)$ is 1/2. The Fourier series of $\theta_\psi(z)$ must start as

$$\theta_\psi(z) = q \pm q^9 \pm q^{25} \pm q^{49} + O(q^{81}).$$

Consequently, with the notation from before, we must have that $a_f = 1$, and from (3.1), that the term $\eta(dz)^{r_d}$ dividing $f(z)$ with smallest d must be either $\eta(8z)$ or $\eta(8z)^{-1}$. We consider only the first case before turning to the general situation of Theorem 1.2.

The Fourier expansion of $\eta_0(8z)$ is given by

$$\eta_0(8z) = 1 - q^8 - q^{16} + O(q^{40}).$$

As before, we now see that $\eta(16z)^{-1}$ must divide $f(z)$, and

$$\frac{\eta_0(8z)}{\eta_0(16z)} = 1 - q^8 - q^{24} + q^{32} + O(q^{40}).$$

Since the modulus of ψ is a power of 2, the proof of Lemma 2.1 implies that we cannot change the coefficient of q^{24}, as it would require the level to be divisible by 3. Of course, this is acceptable, as the coefficient of q^{25} in the expansion of $\theta_\psi(z)$ is permitted to be -1. Continuing, we see that $\eta(32z)$ must divide $f(z)$, which leads us to

$$\frac{\eta_0(8z)\eta_0(32z)}{\eta_0(16z)} = 1 - q^8 - q^{24} + q^{48} + q^{80} - q^{120} - q^{168} + O(q^{224}),$$

a very promising Fourier series. We compute that for $f(z) = \frac{\eta(8z)\eta(32z)}{\eta(16z)}$, we have that $a_f = 1$, $f(z)$ is holomorphic, and $f(z) = \theta_{\chi_8}(z)$ where $\chi_8(\cdot) = \left(\frac{2}{\cdot}\right)$ is a primitive character of conductor 8.

A priori it is conceivable that there are other theta functions expressible as eta-quotients divisible by $\eta(8z)$. However, since we have now reached a Fourier series whose exponents are supported on the squares, if $f(z)$ were additionally divisible by some $\eta(dz)^{r_d}$ with $d > 32$ chosen minimally, then we must have both that d is a power of 2, based on the level, and that d is one less than a square, based on its effect on the Fourier series. But Lemma 2.5 implies that there are no such d, so we have produced the only eta-quotient divisible by $\eta(8z)$ which is a theta function.

We now turn to the proof of Theorem 1.2, assuming that $f(z) = \prod \eta(dz)^{r_d}$ is in Θ_m for some m whose only prime factors are 2 and 3. Under the assumption that $f(z)$ is monic, we must consider two cases: either the Fourier series of $f(z)$ has the form $1 + O(q)$ or it has the form $q + O(q^4)$.

In the first case, the Fourier expansion has the form $1 + O(q)$ and we must have that $m = 1$ since $f(z)$ is a linear combination of theta functions and the only theta function whose Fourier expansions begins in this fashion is $\theta_1(z)$. Lemma 2.1 then tells us that we are limited to factors $\eta(dz)$ with $d \mid 144$. Unfortunately, we must now split into further cases, depending on the location of the first non-zero coefficient after the constant term.

If the Fourier series begins $1 - aq + O(q^2)$ with $a \neq 0$, then we must have that $\eta(z)^a$ is a factor of $f(z)$. We have that

$$\eta_0(z)^a = 1 - aq + \frac{a(a-3)}{2}q^2 + O(q^3),$$

and consequently, we must have that $\eta(2z)^{a_2}$ divides $f(z)$, where $a_2 = \frac{a(a-3)}{2}$. We then see that $\eta(3z)^{a_3}$ must also divide $f(z)$, where $a_3 = \frac{a^3-4a}{3}$. The coefficient of q^4 can be arbitrary, however, so we let $\eta(4z)^b$ denote the power of $\eta(4z)$ dividing $f(z)$. We then compute the coefficient of q^5 in $\eta_0(z)^a\eta_0(2z)^{a_2}\eta_0(3z)^{a_3}\eta_0(4z)^b$ to be

$$a\left(b - \frac{1}{20}a^4 + \frac{3}{4}a^2 - \frac{1}{2}a - \frac{6}{5}\right).$$

This coefficient must be 0, since we cannot ‘fix’ it with some $\eta(dz)$ with $d \mid 144$. Since a was assumed to be non-zero, we see that b is in fact not allowed to be arbitrary. We let a_4 be the required value, namely $\frac{1}{20}a^4 - \frac{3}{4}a^2 + \frac{1}{2}a + \frac{6}{5}$. We then continue as before, seeing that we must have a factor of $\eta(6z)^{a_6}$, where $a_6 = -\frac{1}{30}a^6 - \frac{1}{2}a^3 + \frac{8}{15}a^2 + 2a$. The coefficient of q^7 is then

$$-\frac{2}{35}a(a-1)(a+1)(a-2)(a+2)(a^2+5),$$

and, again since $a \neq 0$, we must have that $a = \pm 1, \pm 2$.

If $a = -2$, the above yields that $f(z)$ is divisible by $\frac{\eta(2z)^5}{\eta(z)^2\eta(4z)^2}$, which is exactly the eta-quotient representation of $\theta_1(z)$ we found earlier. In particular, the exponents in its Fourier series are supported on the squares, and any additional factor $\eta(dz)^{r_d}$ of $f(z)$ must then have the property that d is a square. But because of the $2q$ term in the Fourier series of $f(z)$, this would introduce a term of order q^{d+1} in the Fourier series which we cannot change. Since $d+1$ is not a perfect square, we have found the only possible $f(z)$ with $a = -2$. Similarly, we find that there is exactly one form in Θ_1 for each of $a = -1$, $a = 1$, and $a = 2$:

$$\frac{\eta(2z)^2\eta(3z)}{\eta(z)\eta(6z)}, \ \frac{\eta(z)\eta(4z)\eta(6z)^2}{\eta(2z)\eta(3z)\eta(12z)}, \ \text{and} \ \frac{\eta(z)^2}{\eta(2z)}.$$

If the first non-zero coefficient of $f(z)$ after the constant term is $-aq^4$, the argument above translates almost exactly to this case, with the modification that q is replaced by q^4. One must argue that there can be no $\eta(9z)^{r_9}$ factor, but this is clear, as it would introduce an irreparable q^{13} in the Fourier expansion. We thus obtain in this case the previous eta-quotients with z replaced by $4z$. Two of these are in Θ_1, namely

$$\frac{\eta(8z)^5}{\eta(4z)^2\eta(16z)^2} \ \text{and} \ \frac{\eta(4z)\eta(16z)\eta(24z)^2}{\eta(8z)\eta(12z)\eta(48z)}.$$

If the Fourier series starts $1 - aq^9 + O(q^{16})$, we proceed similarly, obtaining two elements of Θ_1^0, namely $\frac{\eta(9z)^2}{\eta(18z)}$ and $\frac{\eta(18z)^5}{\eta(9z)^2\eta(36z)^2}$. Throughout all of the remaining cases, corresponding to the first non-constant term being q^{16}, q^{36}, and q^{144}, we obtain exactly two forms whose coefficients are supported on the squares: $\frac{\eta(36z)^2}{\eta(72z)}$, which is not in Θ_1, and $\frac{\eta(72z)^5}{\eta(36z)^2\eta(144z)^2}$, which is in Θ_1.

We now consider the case when the Fourier expansion of $f(z)$ has the form $q + O(q^4)$. By Lemma 2.5, the factor $\eta(dz)^{r_d}$ of $f(z)$ with $r_d \neq 0$ and d minimal has $d \in \{3, 8, 24, 48, 288\}$. We consider each of these cases in turn.

If $a := r_3 \neq 0$, in order for the Fourier coefficients to cancel properly, we must also have that $\eta(6z)^{a_2}\eta(9z)^{a_3}\eta(12z)^{a_4}\eta(18z)^{a_5}$ divides $f(z)$, where $a_2 = \frac{a(a-3)}{2}, a_3 = \frac{a(a-2)(a+2)}{3}, a_4 = \frac{a(a+2)(a-1)^2}{4}$, and $a_5 = -\frac{a(a-2)(a+2)(a^3+4a+15)}{30}$. The coefficient of q^{21} in the Fourier expansion of the corresponding $f(z)$ is

$$-\frac{2}{35}a(a-1)(a+1)(a-2)(a+2)(a^2+5),$$

from which we see that a must be one of $\pm1, \pm2$. These give rise to

$$\frac{\eta(3z)^2\eta(12z)^2}{\eta(6z)},\ \frac{\eta(3z)\eta(18z)^2}{\eta(6z)\eta(9z)},\ \frac{\eta(6z)^2\eta(9z)\eta(36z)}{\eta(3z)\eta(12z)\eta(18z)},\ \text{and}\ \frac{\eta(6z)^5}{\eta(3z)^2}.$$

As before, one verifies that each of these is in some Θ_m, and no additional factors $\eta(dz)^{r_d}$ can be added to each of these eta-quotients while maintaining the property that the Fourier coefficients are supported on the squares.

If $r_3 = 0$ and $a := r_8 \neq 0$, then we must also have that $\eta(16z)^{a_2}\eta(24z)^b\eta(32z)^{a_4}$ divides $f(z)$, where $a_2 = \frac{a(a-3)}{2}$, b is arbitrary, and $a_4 = ab - \frac{1}{12}a^4 + \frac{7}{12}a^2 + \frac{1}{2}a$. Requiring the coefficient of q^{40} to be 0 (as we are not permitted to change it), we must have that

$$b = \frac{2a^4 - 20a^2 + 18}{15a}.$$

Since b is an integer, we have that $a \mid 18$, and one observes that the above is an integer only for $a \mid 6$. If $a = \pm1$, we find the two forms

$$\frac{\eta(8z)\eta(32z)}{\eta(16z)}\ \text{and}\ \frac{\eta(16z)^2}{\eta(8z)},$$

both of which are theta functions. Any factors of $\eta(48z)$ or $\eta(288z)$ would introduce irreparable coefficients, so these are all the forms arising from $a = \pm1$. If $a = \pm2$, we find

$$\frac{\eta(8z)^2\eta(48z)}{\eta(16z)\eta(24z)}\ \text{and}\ \frac{\eta(16z)^5\eta(24z)\eta(96z)}{\eta(8z)^2\eta(32z)^2\eta(48z)^2},$$

the first of which is in Θ_2^0, the second of which is also a form of some interest, namely $\theta_\chi(z) + 3\theta_\chi(9z)$ where $\chi(n) = \left(\frac{2}{n}\right)$, but the Serre-Stark basis theorem [13] implies that it cannot lie in any Θ_m. If $a = \pm3$, we obtain the forms

$$\eta(8z)^3\ \text{and}\ \frac{\eta(16z)^9}{\eta(8z)^3\eta(32z)^3},$$

both of which are theta functions. Lastly, if $a = \pm6$, we obtain no forms, eventually running into a non-zero coefficient of q^{88}.

We now suppose that $\eta(24z)^a\eta(48z)^b$ is the smallest divisor of $f(z)$. We consider these two variables, since either coefficient can be arbitrary. Consequently, we permit one, but not both, of a and b to be 0. We proceed as before, eventually finding potentially problematic coefficients of q^{240} and q^{264}, say A_1 and A_2, respectively. Both A_1 and A_2 are polynomials in a and b, whose degrees in a are 10 and 11, respectively, and whose degrees in b are both 5. A_1 is irreducible, whereas A_2 has a factor of a and is otherwise irreducible. Substituting $a = 0$ into A_1, we find that we must have $b(b^4 - 6) = 0$, which cannot hold since we assumed that not both a and b are 0. Now, we compute the resultant in b of A_1 and A_2/a. This yields a degree 50 polynomial in a whose only rational roots are $a = 1, a = -1, a = 5$, and $a = -5$, which yield that $b = 0$ or -2, $b = 1$ or 3, $b = -2$, and $b = 13$, respectively. These yield the forms $\eta(24z)$, $\frac{\eta(24z)\eta(96z)\eta(144z)^5}{\eta(48z)^2\eta(72z)^2\eta(288z)^2}$, $\frac{\eta(48z)\eta(72z)^2}{\eta(24z)\eta(144z)}$, $\frac{\eta(48z)^3}{\eta(24z)\eta(96z)}$, $\frac{\eta(24z)^5}{\eta(48z)^2}$, and $\frac{\eta(48z)^{13}}{\eta(24z)^5\eta(96z)^5}$.

Finally, we suppose that $\eta(288z)^a$ is the smallest divisor of $f(z)$. We then must also have that

$$\eta(576z)^{a_2}\eta(864z)^{a_3}\eta(1152z)^{a_4}$$

divides $f(z)$, where $a_2 = \frac{a(a-3)}{2}$, $a_3 = \frac{a(a-2)(a+2)}{3}$, and $a_4 = \frac{a(a+2)(a-1)^2}{4}$. The coefficient of q^{1440} is $\frac{1}{5}a(a^4 - 6)$, which is non-zero, and so there are no such $f(z)$. This finishes the proof of Theorem 1.2.

References

[1] V. Blomer and G. Harcos. Hybrid bounds for twisted L-functions. *J. Reine Angew. Math.*, 621:53–79, 2008.

[2] K. Bringmann and K. Ono. Coefficients of harmonic weak maass forms. *Proc. of the 2008 Univ. of Flor. Conf.*, accepted for publication.

[3] J. H. Bruinier, G. van der Geer, G. Harder, and D. Zagier. *The 1-2-3 of modular forms*. Universitext. Springer-Verlag, Berlin, 2008. Lectures from the Summer School on Modular Forms and their Applications held in Nordfjordeid, June 2004, Edited by Kristian Ranestad.

[4] F. Diamond and J. Shurman. *A first course in modular forms*, volume 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[5] D. Dummit, H. Kisilevsky, and J. McKay. Multiplicative products of η-functions. In *Finite groups—coming of age (Montreal, Que., 1982)*, volume 45 of *Contemp. Math.*, pages 89–98. Amer. Math. Soc., Providence, RI, 1985.

[6] T. Jagathesan and M. Manickam. On Shimura correspondence for non-cusp forms of half-integral weight. *J. Ramanujan Math. Soc.*, 23(3):211–222, 2008.

[7] Y. Martin. Multiplicative η-quotients. *Trans. Amer. Math. Soc.*, 348(12):4825–4856, 1996.

[8] G. Mersmann. Holomorphe η-produkte und nichtverschwindende ganze modulformen für $\gamma_0(n)$. *Master's Thesis, Univ. of Bonn*, 1991.

[9] P. Mihăilescu. Primary cyclotomic units and a proof of Catalan's conjecture. *J. Reine Angew. Math.*, 572:167–195, 2004.

[10] K. Ono. *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, volume 102 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2004.

[11] H. Rademacher. On the expansion of the partition function in a series. *Ann. of Math. (2)*, 44:416–422, 1943.

[12] H. Rademacher and H. S. Zuckerman. On the Fourier coefficients of certain modular forms of positive dimension. *Ann. of Math. (2)*, 39(2):433–462, 1938.

[13] J.-P. Serre and H. M. Stark. Modular forms of weight 1/2. In *Modular functions of one variable, VI (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976)*, pages 27–67. Lecture Notes in Math., Vol. 627. Springer, Berlin, 1977.

[14] H. S. Zuckerman. On the coefficients of certain modular forms belonging to subgroups of the modular group. *Trans. Amer. Math. Soc.*, 45(2):298–321, 1939.

[15] H. S. Zuckerman. On the expansions of certain modular forms of positive dimension. *Amer. J. Math.*, 62:127–152, 1940.

Emory University, Department of Mathematics and Computer Science, 400 Dowman Dr, Atlanta, GA 30322