EFFECTIVE LOG-FREE ZERO DENSITY ESTIMATES FOR AUTOMORPHIC L-FUNCTIONS AND THE SATO-TATE CONJECTURE

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Abstract. Let $K/\mathbb{Q}$ be a number field. Let π and π' be cuspidal automorphic representations of $\mathrm{GL}_d(\mathbb{A}_K)$ and $\mathrm{GL}_{d'}(\mathbb{A}_K)$. We prove an unconditional and effective log-free zero density estimate for all automorphic L-functions $L(s,\pi,K)$ and prove a similar estimate for Rankin-Selberg L-functions $L(s,\pi\otimes\pi',K)$ when π or π' satisfies the Ramanujan conjecture. As applications, we make effective Moreno's analogue of Hoheisel's short interval prime number theorem and extend it to the context of the Sato-Tate conjecture; additionally, we bound the least prime in the Sato-Tate conjecture in analogy with Linnik's theorem on the least prime in an arithmetic progression. We also prove effective log-free density estimates for automorphic L-functions averaged over twists by Dirichlet characters and consider and prove an "average Hoheisel" result for GL_2 L-functions.

1. Introduction and statement of results

The classical prime number theorem asserts that

$$\sum_{n\le x}\Lambda(n)\sim x,$$

where $\Lambda(n)$ is the von Mangoldt function. Depending on the quality of the error term, it is possible to deduce from this a prime number theorem for short intervals, in the form

$$\sum_{x<n\le x+h}\Lambda(n)\sim h, \tag{1.1}$$

provided that h is not too small; with the presently best known error terms, we may take h a bit smaller than x divided by any power of $\log x$, but not as small as $x^{1-\delta}$ for any $\delta>0$. Improving the error bound in the prime number theorem to allow for h to be of size $x^{1-\delta}$ is a monumentally hard task, known as the quasi-Riemann hypothesis, and amounts to showing that there are no zeros of the Riemann zeta function $\zeta(s)$ in the region $\mathrm{Re}(s)>1-\delta$. Nevertheless, in 1930, Hoheisel [23] made the remarkable observation that, with Littlewood's improved zero-free region for $\zeta(s)$, if there are simply *not too many* zeros in this region, then one can deduce (1.1) with $h=x^{1-\delta}$. In particular, it turns out that

$$N(\sigma,T):=\#\{\rho=\beta+i\gamma\colon \zeta(\rho)=0,\beta\ge\sigma,|\gamma|\le T\}\ll T^{c(1-\sigma)}(\log T)^{c'}, \tag{1.2}$$

where $c>2$ and $c'>0$ are absolute constants; this is a so-called **zero density estimate**. (In this section, c and c' will always denote positive absolute constants, though they may represent different values in each occurrence.) Recall that there are about $\frac{T}{\pi}\log\frac{T}{2\pi e}$ zeros of $\zeta(s)$ with $|\gamma|\le T$, so that a vanishingly small proportion of zeros have real part close to 1.

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An explicit version of (1.2) enabled Hoheisel to prove the prime number theorem in short intervals (1.1) for $h = x^{1-\delta}$ in the range $0 \le \delta \le 1/33000$; it is now known that we may take $0 \le \delta \le \frac{5}{12}$, due to Huxley [24] and Heath-Brown [20].

Another classical problem in analytic number theory is to determine the least prime in an arithmetic progression $a \pmod q$ with $(a, q) = 1$. Linnik [32] was able to show that the least such prime is no bigger than q^A, where A is an absolute constant; the best known value of A is 5, due to Xylouris [54] in his Ph.D. thesis. Modern treatments of Linnik's theorem typically use a simplification due to Fogels [13], which involves proving a more general version of (1.2) for Dirichlet L-functions $L(s,\chi)$. Specifically, if we define

$$N_\chi(\sigma, T) := \#\{\rho = \beta + i\gamma\colon L(\rho,\chi) = 0, \beta \ge \sigma, \text{ and } |\gamma| \le T\},$$

then Fogels showed that

$$\sum_{\chi \pmod q} N_\chi(\sigma, T) \ll T^{c(1-\sigma)} \tag{1.3}$$

when $T \ge q$. Due to the absence of a $\log T$ term as compared to (1.2), it is standard to call such a result a **log-free zero density estimate**. In this paper, we are interested in analogous log-free zero density estimates for automorphic L-functions and their arithmetic applications, specifically to analogues of Hoheisel's and Linnik's theorems.

We consider the following general setup. Let $K/\mathbb{Q}$ be a number field with ring of adeles $\mathbb{A}_K$, and let $\mathcal{A}_d(K)$ denote the set of all cuspidal automorphic representations of $\mathrm{GL}_d(\mathbb{A}_K)$ with unitary central character. We make the implicit assumption that the central character of π is trivial on the product of positive reals when embedded diagonally into the (archimedean places of) the ideles. If $\pi \in \mathcal{A}_d(K)$, then there is an L-function $L(s, \pi, K)$ attached to π whose Dirichlet series and Euler product are given by

$$L(s, \pi, K) = \sum_{\mathfrak{a}} \frac{\lambda_\pi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s} = \prod_{\mathfrak{p}} \prod_{j=1}^{d} (1 - \alpha_\pi(j, \mathfrak{p})\mathrm{N}\mathfrak{p}^{-s})^{-1},$$

where the sum runs over the non-zero integral ideals of K, the product runs over the prime ideals of K, and $\mathrm{N}\mathfrak{a} = \mathrm{N}_{K/\mathbb{Q}}\mathfrak{a}$ denotes the norm of the ideal $\mathfrak{a}$.

Let $\pi \in \mathcal{A}_d(K)$ and $\pi' \in \mathcal{A}_{d'}(K)$. The Rankin-Selberg L-function

$$L(s, \pi \otimes \pi', K) = \sum_{\mathfrak{a}} \frac{\lambda_{\pi\otimes\pi'}(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s} \doteq \prod_{\mathfrak{p}} \prod_{j_1=1}^{d} \prod_{j_2=1}^{d'} (1 - \alpha_\pi(j_1, \mathfrak{p})\alpha_{\pi'}(j_2, \mathfrak{p})\mathrm{N}\mathfrak{p}^{-s})^{-1}$$

also has an analytic continuation and a functional equation; here, the symbol $\doteq$ indicates equality up to a contribution from the finitely many ramified prime ideals, whose Euler factors are slightly more complicated. Under our normalization for the central characters of π and π', the Rankin-Selberg L-function $L(s, \pi \otimes \pi', K)$ has a pole of order one at $s = 1$ if and only if $\pi' \cong \tilde{\pi}$.

Define $\Lambda_{\pi\otimes\pi'}(\mathfrak{a})$ by the Dirichlet series identity

$$-\frac{L'}{L}(s, \pi \otimes \pi', K) = \sum_{\mathfrak{a}} \frac{\Lambda_{\pi\otimes\pi'}(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s}.$$

If $\tilde{\pi}$ is the representation which is contragredient to π, then it follows from standard Rankin-Selberg theory and the Wiener-Ikehara Tauberian theorem that we have a prime number theorem for $L(s,\pi\otimes\tilde{\pi},K)$ in the form

$$\sum_{\mathrm{N}\mathfrak{a}\leq x}\Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{a})\sim x.$$

It is reasonable to expect (for example, it follows from the generalized Riemann hypothesis) that there is some small $\delta>0$ such that for x sufficiently large and any $h\geq x^{1-\delta}$, we have

$$\sum_{x<\mathrm{N}\mathfrak{a}\leq x+h}\Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{a})\sim h. \tag{1.4}$$

Unfortunately, a uniform analogue of Littlewood's improved zero-free region does not yet exist for all automorphic L-functions, so it seems that (1.4) is currently inaccessible except in special situations. However, it follows from the work of Moreno [38] that if $L(s,\pi\otimes\tilde{\pi},K)$ has a so-called "standard" zero-free region (one of a quality similar to Hadamard's and de la Vallée Poussin's for $\zeta(s)$), and if there is a log-free zero density estimate of the form

$$N_{\pi\otimes\pi'}(\sigma,T):=\#\{\rho=\beta+i\gamma\colon L(\rho,\pi\otimes\pi',K)=0,\beta\geq\sigma,|\gamma|\leq T\}\ll T^{c_{\pi,\pi'}(1-\sigma)}$$

for $L(s,\pi\otimes\tilde{\pi},K)$, then for any $0<\delta<1/c_{\pi,\tilde{\pi}}$ and any $h\geq x^{1-\delta}$, one has

$$\sum_{x<\mathrm{N}\mathfrak{a}\leq x+h}\Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{a})\gg h, \tag{1.5}$$

which Moreno called the **Hoheisel phenomenon**. However, at the time of Moreno's work, such log-free zero density estimates only existed in special cases. Moreover, in general, it is only known that $L(s,\pi\otimes\tilde{\pi},K)$ has a standard zero-free region if π is self-dual.

Effective log-free zero density estimates have been proven for certain natural families of L-functions. Weiss [52] proved an effective analogue of (1.3) for the Hecke L-functions of ray class characters, which enabled him to access prime ideals of K satisfying splitting conditions in a finite Galois extension M/K. Additionally, Kowalski and Michel [30] obtained a log-free zero density estimate in the conductor aspect for L-functions associated to any family of automorphic representations of $\mathrm{GL}_d(\mathbb{A}_{\mathbb{Q}})$ satisfying certain conditions, including the generalized Ramanujan conjecture (see Section 2.1). Their result works best when the T-aspect is essentially irrelevant (see [30, Remark 3]), which is useful for establishing variants of Linnik's theorem but not (1.5).

Recall that $\pi\in\mathcal{A}_d(K)$ and $\pi'\in\mathcal{A}_{d'}(K)$. Building on Fogels's log-free zero density estimate for Dirichlet L-functions, Akbary and Trudgian [1] proved that if $K=\mathbb{Q}$, either $\max\{d,d'\}\leq 2$ or one of π and π' is self-dual, T is sufficiently large in terms of π and π', and there exists a constant $0<\delta<1$ such that

$$\sum_{x<n\leq x+x^{1-\delta}}\Lambda_{\pi\otimes\tilde{\pi}}(n)\ll_{\pi}x^{1-\delta}\quad\text{and}\quad\sum_{x<n\leq x+x^{1-\delta}}\Lambda_{\pi'\otimes\tilde{\pi}'}(n)\ll_{\pi'}x^{1-\delta}, \tag{1.6}$$

then

$$N_{\pi\otimes\pi'}(\sigma,T)\leq T^{c_{d,d'}(1-\sigma)},$$

where $c_{d,d'}>0$ is a constant depending on d and d'. By the work of Moreno mentioned earlier, this allowed them to establish a variant of (1.5) for $\pi\otimes\tilde{\pi}$ when π is self-dual, provided that π satisfies (1.6). The condition (1.6) follows immediately from the Brun-Titchmarsh theorem [37], provided that the generalized Ramanujan conjecture holds for $L(s,\pi,\mathbb{Q})$. Thus, their work is unconditional in many cases of interest.

However, even in cases where the work of Akbary and Trudgian is unconditional, they do not make clear the dependence of the allowable range of T on π and π', which is necessary to obtain analogues of Linnik's theorem. Moreover, the length of the allowable intervals in the Hoheisel phenomenon depends on the constant $c_{d,d'}$, and Akbary and Trudgian do not make clear its dependence on d and d'. This makes the density estimate in [1] difficult to use in situations where uniformity in parameters over several L-functions is relevant, especially when the L-functions in question vary in degree. Moreover, as mentioned earlier, the method of [1] builds on work of Fogels, which, as Weiss mentions in the introduction of his Ph.D. thesis [53] regarding GL_1 L-functions, will produce an undesireable dependence on π and π'.

The main goals of this paper are to prove several log-free zero density estimates for Rankin-Selberg L-functions $L(s,\pi\otimes\pi',K)$, where the dependence on all parameters is made effective, and to derive arithmetic corollaries along the lines of the Hoheisel phenomenon and Linnik's theorem for which this effectivity is crucial. These results, which hold independently of whether π and π' are self-dual, are most naturally stated in terms of the analytic conductors $\mathfrak{q}(\pi)$ and $\mathfrak{q}(\pi')$, whose definitions we give in (2.3). We begin with the following theorem.

Theorem 1.1. *Let K be a number field, let $\pi\in\mathcal{A}_d(K)$, and let $\pi'\in\mathcal{A}_{d'}(K)$. Suppose that π' satisfies the generalized Ramanujan conjecture (GRC), and let $T\geq[K:\mathbb{Q}]$. There exists an absolute constant $c_1>0$ such that if $\frac{1}{2}\leq\sigma\leq 1$, then*[1]

$$N_{\pi\otimes\pi'}(\sigma,T)\ll(d')^2(\mathfrak{q}(\pi)\mathfrak{q}(\pi')T^{[K:\mathbb{Q}]})^{c_1 d' d(1-\sigma)}.$$

The next result, which is unconditional, follows from the proof of Theorem 1.1 by letting π' be the trivial representation of $\mathrm{GL}_1(\mathbb{A}_K)$, which visibly satisfies GRC.

Corollary 1.2. *Let K be a number field, and let $\pi\in\mathcal{A}_d(K)$. If $T\geq[K:\mathbb{Q}]$ and $\frac{1}{2}\leq\sigma\leq 1$, then*

$$N_\pi(\sigma,T)\ll(\mathfrak{q}(\pi)T^{[K:\mathbb{Q}]})^{c_1 d(1-\sigma)}.$$

Remark. Corollary 1.2 is the first unconditional log-free zero density estimate for all automorphic L-functions $L(s,\pi,K)$. (Recall that Akbary and Trudgian's result is conditional on the verification of (1.6) for *both* π and π'. Thus, even if π' is trivial, their result is still conditional on $L(s,\pi,\mathbb{Q})$ satisfying (1.6).) Moreover, Corollary 1.2 applies to L-functions over any number field, whereas Akbary and Trudgian considered only L-functions over $\mathbb{Q}$.

In addition to density estimates of the form (1.3), Jutila [26] and Montgomery [35] proved "hybrid" density estimates of the form

$$\sum_{q\leq Q}\ \sideset{}{^\star}\sum_{\chi \bmod q} N_\chi(\sigma,T)\ll(Q^2T)^{c(1-\sigma)}(\log QT)^{c'}, \tag{1.7}$$

where the $\star$ on the summation indicates it is to be taken over primitive characters. (By [35], we may take $c=\frac{5}{2}$.) This simultaneously generalizes (1.2) and Bombieri's large sieve density estimate [7]. As a consequence of (1.7), one sees that the average value of $N_\chi(\sigma,T)$ is noticeably smaller than what is given by (1.3). Furthermore, (1.7) can be used to prove versions of the Bombieri-Vinogradov theorem in both long and short intervals.

Gallagher [15] proved that

$$\sum_{q\leq T}\ \sideset{}{^\star}\sum_{\chi \bmod q} N_\chi(\sigma,T)\ll T^{c(1-\sigma)},\qquad T\geq 1, \tag{1.8}$$

[1]Unless mentioned otherwise, the implied constant in an asymptotic inequality is absolute and computable.

providing a mutual refinement of (1.3) and (1.7). Our second result generalizes (1.8) to consider twists of Rankin-Selberg L-functions associated to automorphic representations over $\mathbb{Q}$.

Theorem 1.3. *Under the notation and hypotheses of Theorem 1.1 with $K = \mathbb{Q}$,*

$$\sum_{\substack{q \leq T \\ \gcd(q, q(\pi)q(\pi'))=1}} \sideset{}{^\star}\sum_{\chi \bmod q} N_{(\pi\otimes\chi)\otimes\pi'}(\sigma, T) \ll (d')^2(\mathfrak{q}(\pi)\mathfrak{q}(\pi')T)^{c_1 d' d(1-\sigma)},$$

where $q(\pi)$ and $q(\pi')$ are the conductors of π and π', respectively.

As with Theorem 1.1, we immediately obtain the following unconditional corollary by letting π' be the trivial representation of $\mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$.

Corollary 1.4. *Under the notation and hypotheses of Corollary 1.2 with $K = \mathbb{Q}$,*

$$\sum_{\substack{q \leq T \\ \gcd(q, q(\pi))=1}} \sideset{}{^\star}\sum_{\chi \bmod q} N_{\pi\otimes\chi}(\sigma, T) \ll (\mathfrak{q}(\pi)T)^{c_1 d(1-\sigma)}.$$

It follows from Theorem 1.1 and Corollary 1.2 that we obtain log-free zero density estimates for Rankin-Selberg L-functions which factor as a product of L-functions, each of which individually satisfies the hypotheses of Theorem 1.1 or Corollary 1.2. The Langlands principle of functoriality predicts that such a factorization always exists for $L(s, \pi\otimes\pi', K)$, and this is known to be true in certain cases. For example, this is known when $\pi, \pi' \in \mathcal{A}_2(K)$ by work of Ramakrishnan [45, Theorem M]. This is also known when $\pi \in \mathcal{A}_2(K)$ and $\pi' \in \mathcal{A}_3(K)$ by work of Kim and Shahidi [29] and Ramakrishnan and Wang [47]. This allows us to deduce the following unconditional result.

Theorem 1.5. *Let K be a number field. Let $\pi \in \mathcal{A}_d(K)$ and $\pi' \in \mathcal{A}_{d'}(K)$ with $d \leq 2$ and $d' \leq 3$. If $T \geq [K : \mathbb{Q}]$ and $\frac{1}{2} \leq \sigma \leq 1$, then*

$$N_{\pi\otimes\pi'}(\sigma, T) \ll (\mathfrak{q}(\pi)\mathfrak{q}(\pi')T^{[K:\mathbb{Q}]})^{6c_1(1-\sigma)}.$$

If $K = \mathbb{Q}$, then

$$\sum_{\substack{q \leq T \\ \gcd(q, q(\pi)q(\pi'))=1}} \sideset{}{^\star}\sum_{\chi \bmod q} N_{(\pi\otimes\chi)\otimes\pi'}(\sigma, T) \ll (\mathfrak{q}(\pi)\mathfrak{q}(\pi')T)^{6c_1(1-\sigma)}.$$

In particular, Theorem 1.5 applies to $L(s, \pi\otimes\pi', K)$ when $\pi, \pi' \in \mathcal{A}_2(K)$ each correspond to Hecke-Maass forms for which GRC is not known. The special case where $K = \mathbb{Q}$, π corresponds to a Hecke-Maass form, and $\pi' \cong \tilde{\pi}$ was proved by Motohashi [42] using methods different from our own.

Another example of an L-function for which GRC is not known but there is a profitable factorization is $L(s, \mathrm{Sym}^2\pi\otimes\mathrm{Sym}^2\pi, K)$, where $\pi \in \mathcal{A}_2(K)$ is associated to a self-dual Hecke-Maass form. Speficically, we have the well-known factorization

$$L(s, \mathrm{Sym}^2\pi \otimes \mathrm{Sym}^2\pi, K) = \zeta_K(s)L(s, \mathrm{Sym}^2\pi, K)L(s, \mathrm{Sym}^4\pi, K),$$

and it is known by work of Gelbart and Jacquet [16] and Kim [27] that $\mathrm{Sym}^2\pi \in \mathcal{A}_3(K)$ and $\mathrm{Sym}^4\pi \in \mathcal{A}_5(K)$, respectively. Thus we obtain an unconditional log-free zero density estimate for $L(s, \mathrm{Sym}^2\pi \otimes \mathrm{Sym}^2\pi, K)$ from three applications of Corollary 1.2.

1.1. **Arithmetic applications.** We now turn to the applications of Theorems 1.1 and 1.3 and their corollaries. We begin by considering a version of (1.5) with effective bounds on the size of the intervals for all L-functions $L(s,\pi\otimes\tilde{\pi},K)$ satisfying the generalized Ramanujan conjecture (not just those with a standard zero-free region).

Theorem 1.6. *Assume the above notation. Let $\pi\in\mathcal{A}_d(K)$ satisfy GRC. There exists an absolute constant $0<c_2<1/2$ such that if*

$$\delta\leq\frac{c_2}{d^2[K:\mathbb{Q}]\log(3d)},$$

x is sufficiently large (with respect to π), and $h\geq x^{1-\delta}$, then

$$\sum_{x<\mathrm{N}\mathfrak{a}\leq x+h}\Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{a})\asymp h,$$

where the implied constant depends on π and K. Moreover, if $L(s,\pi\otimes\tilde{\pi},K)$ factors as a product of L-functions satisfying the hypotheses of Theorem 1.1 or Corollary 1.2, then the result is not conditional on GRC. In particular, if $d=2$, then the result is unconditional.

Remark. When $L(s,\pi\otimes\tilde{\pi},K)$ factors as a product of L-functions of cuspidal automorphic representations, then Theorem 1.6 *confirms* the hypothesis (1.6) of Akbary and Trudgian's work. This is particularly interesting when π is associated to a Hecke-Maass form over K, where GRC is not known. However, Akbary and Trudgian are concerned only with the case $K=\mathbb{Q}$, and Motohashi [42] recently proved a version of Theorem 1.6 using his aforementioned log-free zero density estimate.

It is of course somewhat unsatisfying that we are not able to obtain an asymptotic formula in Theorem 1.6 to provide a true short interval analogue of (1.5). As remarked earlier, this is due to the lack of a strong zero-free region for general automorphic L-functions and seems unavoidable at present. Good zero-free regions of a quality better than Littlewood's exist for Dedekind zeta functions (for example, due to Mitsui [34]), which enabled Balog and Ono [2] to prove a prime number theorem for prime ideals in Chebotarev sets lying in short intervals.

Even though versions of Theorem 1.6 with asymptotic equality are only known in special cases, we can use Theorem 1.3 to show that the predicted asymptotic holds on average. Our motivation is a result of Gallagher [15, Theorem 7], which states that there exist absolute constants $c_3,c_4\in(0,1)$ such that if $\exp(\sqrt{\log x})\leq Q\leq x^{c_3}$ and $x/Q\leq h\leq x$, then

$$\sum_{q\leq Q}\sideset{}{^\star}\sum_{\chi\bmod q}\Big|\sum_{x<p\leq x+h}\Lambda(p)\chi(p)-\delta(\chi)h+\delta_{q,*}(\chi)h\xi^{\beta_1-1}\Big|\ll h\exp\Big(-\frac{c_4\log x}{\log Q}\Big)\tag{1.9}$$

for some $\xi\in[x,x+h]$. Here, $\delta(\chi)=1$ if χ is the trivial character and is zero otherwise, and β_1 denotes the Landau-Siegel zero associated to an exceptional real Dirichlet character $\chi^*\ (\mathrm{mod}\ q)$ if it exists. We set $\delta_{q,*}(\chi)=1$ if $\chi=\chi^*$ and zero otherwise.

We prove the following generalization of (1.9); to obtain unconditional results and simplify the exposition, we restrict ourselves to consider automorphic representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ with trivial central character.

Theorem 1.7. *Assume the above notation. Let $\pi\in\mathcal{A}_2(\mathbb{Q})$ have a trivial central character. There exist absolute constants $c_3,c_4\in(0,1)$ such that if $\exp(\sqrt{\log x})\leq Q\leq x^{c_3}$ and $x/Q\leq$*

$h \leq x$, then

$$\sum_{\substack{q \leq Q \\ \gcd(q, q(\pi))=1}} \sideset{}{^\star}\sum_{\chi \bmod q} \Big| \sum_{x < p \leq x+h} \Lambda_{\pi\otimes\tilde{\pi}}(p)\chi(p) - \delta(\chi)h + \delta_{q,*}(\chi) h\xi^{\beta_1 - 1} \Big| \ll h \exp\Big(- \frac{c_4 \log x}{\log(Q\mathfrak{q}(\pi))}\Big)$$

for some $\xi \in [x, x+h]$. The implied constant depends at most on $\mathfrak{q}(\pi)$.

Unlike the previous log-free zero density estimates for general automorphic L-functions discussed earlier, Theorem 1.1 allows us to handle questions where maintaining uniformity in parameters is crucial. One famous example of such an application is the Sato-Tate conjecture, which concerns the distribution of the quantities $\lambda_\pi(\mathfrak{p})$ attached to representations $\pi \in \mathcal{A}_2(K)$, where K is a totally real field; for generalizations to higher degree representations, see, for example, Serre [49]. Suppose that π has trivial central character and is *genuine* (see Section 5.3 for a definition). Suppose further that π satisfies GRC. Then $|\lambda_\pi(\mathfrak{p})| \leq 2$ at all unramified $\mathfrak{p}$. We may thus write $\lambda_\pi(\mathfrak{p}) = 2\cos\theta_\mathfrak{p}$ for some angle $\theta_\mathfrak{p} \in [0, \pi]$. The Sato-Tate conjecture predicts that if $I = [a, b] \subset [-1, 1]$ is a fixed subinterval, then

$$\lim_{x\to\infty} \frac{1}{\pi_K(x)} \#\{\mathrm{N}\mathfrak{p} \leq x \colon \cos\theta_\mathfrak{p} \in I\} = \frac{2}{\pi}\int_I \sqrt{1-t^2}\, dt =: \mu_{\mathrm{ST}}(I),$$

where $\pi_K(x) := \#\{\mathfrak{p} \colon \mathrm{N}\mathfrak{p} \leq x\}$. The Sato-Tate conjecture is now a theorem for large classes of π. For newforms over $\mathbb{Q}$ and elliptic curves over totally real fields, this was proved by Barnet-Lamb, Geraghty, Harris, and Taylor [5], and for Hilbert modular forms, this was done by Barnet-Lamb, Gee, and Geraghty [4]. The proofs rely upon showing that the symmetric power L-functions $L(s, \mathrm{Sym}^n\pi, K)$ are all potentially automorphic, that is, there exists a finite, totally real Galois extension L/K such that $\mathrm{Sym}^n\pi$ is automorphic over L. It is expected that $L(s, \mathrm{Sym}^n\pi, K) \in \mathcal{A}_{n+1}(K)$ for each $n \geq 1$, but as of right now, this is known in general only for $n \leq 4$ [16, 27, 28, 29]. By recent work of Clozel and Thorne [9], if π is associated to a Hilbert modular form, and $K \cap \mathbb{Q}(e^{2\pi i/35}) = \mathbb{Q}$, then $L(s, \mathrm{Sym}^n\pi, K) \in \mathcal{A}_{n+1}(K)$ for $n \leq 8$. Despite this recent progress, because of our limited knowledge of automorphy, the number of symmetric powers needed to access the interval I is particularly important in the sorts of analytic problems considered in this paper.

Recall that the Chebyshev polynomials $U_n(t)$, defined by

$$\sum_{n=0}^{\infty} U_n(t)x^n = \frac{1}{1 - 2tx + x^2},$$

form an orthnormal basis for $L^2([-1, 1], \mu_{\mathrm{ST}})$. If $\pi_\mathfrak{p}$ is unramified, then $U_n(\cos\theta_\mathfrak{p})$ is the Dirichlet coefficient of $L(s, \mathrm{Sym}^n\pi, K)$ at the prime $\mathfrak{p}$. We say that a subset $I \subseteq [-1, 1]$ can be **Sym$^{\mathrm{N}}$-minorized** if there exist constants $b_0, \ldots, b_N \in \mathbb{R}$ with $b_0 > 0$ such that

$$\mathbf{1}_I(t) \geq \sum_{n=0}^{N} b_n U_n(t) \tag{1.10}$$

for all $t \in [-1, 1]$, where $\mathbf{1}_I(\cdot)$ denotes the indicator function of I. Note that if I can be Sym^N-minorized, then it is the union of intervals which individually need not be Sym^N-minorizable. We prove the following result.

Theorem 1.8. *Assume the above notation. Let K be a totally real number field, and let $\pi \in \mathcal{A}_2(K)$ be genuine. Suppose that π satisfies GRC and has trivial central character.*

Suppose that a fixed subset $I \subseteq [-1,1]$ can be Sym^N*-minorized and that* $\mathrm{Sym}^n\pi \in \mathcal{A}_{n+1}(K)$ *for each $n \leq N$. Let $B = \max_{0\leq n\leq N} |b_n|/b_0$, where $b_0, \dots, b_N$ are as in* (1.10)*. There exists an absolute constant $c_5 > 0$ such that if*

$$\delta \leq \frac{c_5}{N[K:\mathbb{Q}]\log(3BN)},$$

x is sufficiently large (with respect to π and N), and $h \geq x^{1-\delta}$, then

$$\sum_{\substack{x<\mathrm{N}\mathfrak{p}\leq x+h \\ \pi_{\mathfrak{p}} \text{ unramified}}} \mathbf{1}_I(\cos\theta_{\mathfrak{p}})\log \mathrm{N}\mathfrak{p} \asymp h,$$

where the implied constant depends on B, I, and K. In particular, if I can be Sym^4*-minorized, or if I can be* Sym^8*-minorized and π is a Hecke newform over $\mathbb{Q}$, then this is unconditional.*

Remarks. 1. For any fixed n, determining the intervals I that can be Sym^N-minorized is an elementary combinatorial problem. We carry this out in Lemma A.1 to determine the intervals that can be Sym^4-minorized, which we consider to be the most interesting case; it turns out that the proportion of subintervals of $[-1,1]$ which can be Sym^4-minorized is roughly 0.388. If one is not concerned with obtaining the optimal minorant or if N is large, it is likely more convenient to apply a standard minorant for I instead. For the Beurling-Selberg minorant (see Montgomery [36, Lecture 1]), a tedious calculation shows that if $N \geq 3$ and $\mu_{\mathrm{ST}}(I) \geq 4(1+\delta)/(N+1)$ for some $\delta > 0$, then I can be Sym^N-minorized with $B \leq \frac{3}{4}(N+1)\delta^{-1}$. It follows that any interval can be Sym^N-minorized for N sufficiently large, and thus every interval is at least conditionally covered by Theorem 1.8. However, Lemma A.2 shows that this minorant might be far from optimal. With the Beurling-Selberg minorant, we prove unconditional results for intervals I satisfying $\mu_{\mathrm{ST}}(I) > \frac{4}{5}$. By contrast, Lemma A.1 implies unconditional results for all intervals satisfying $\mu_{\mathrm{ST}}(I) \geq 0.534$, and for some with measure as small as 0.139.

2. It is tempting to ask whether one can exploit existing results on potential automorphy for symmetric power L-functions and the explicit dependence on the base field in Theorem 1.1 to obtain unconditional, albeit weaker, results for all subintervals of $[-1,1]$. The proof of the Sato-Tate conjecture crucially relies on the work of Moret-Bailly [41] establishing the existence of number fields over which certain varieties have points. The proof of this result unfortunately only permits control over the ramification at finitely many places, so it is not possible to even obtain bounds on the discriminants of the fields over which the symmetric power L-functions are automorphic. Thus, the authors do not believe it is possible to obtain an unconditional analogue of Theorem 1.8 for all I at this time.

As mentioned earlier, Theorem 1.1 also allows us to access Linnik-type questions. As one such example, we consider an analogue of Linnik's theorem in the context of the Sato-Tate conjecture. One complication in the proof of Linnik's theorem that is not seen in Hoheisel's is the possible existence of a so-called Landau-Siegel zero for some Dirichlet L-function $L(s,\chi)$. In order to handle this possible contribution (as one must, since Linnik's theorem is unconditional), two facts are used: there is at most one character $\chi \pmod{q}$ whose associated L-function has a Landau-Siegel zero, and every coefficient in the $\pmod{q}$ Fourier decomposition of the indicator function of set $\{n \in \mathbb{Z} \colon n \equiv a \pmod{q}\}$ is of the same size. Neither of these facts need be true for symmetric power L-functions $L(s, \mathrm{Sym}^n\pi, K)$ and the minorant (1.10), so we consequently say that the minorant (1.10) **does not admit Landau-Siegel zeros** if for every $1 \leq n \leq N$ for which $L(s, \mathrm{Sym}^n\pi, K)$ has a Landau-Siegel zero, the coefficient b_n satisfies $b_n \leq 0$. (It happens that if $b_n \leq 0$, then the Landau-Siegel zero may be trivially ignored in the analysis.) Finally, if a set $I \subseteq [-1, 1]$ admits such a minorant, then we say that I can be Sym^N-minorized without admitting Landau-Siegel zeros. We have suppressed the role of the representation π in this terminology, but its presence will always be clear in context.

Theorem 1.9. *Assume the notation of Theorem 1.8, and in particular that* $I \subset [-1, 1]$ *can be* Sym^N*-minorized. Let* $\pi \in \mathcal{A}_2(K)$ *satisfy the hypotheses of Theorem 1.8. Suppose further that the* Sym^N*-minorant admits no Landau-Siegel zeros. If the Dedekind zeta function* $\zeta_K(s)$ *has no Landau-Siegel zero, then there exists an absolute constant* $c_6 > 0$ *such that if* $\mathrm{Sym}^n\pi \in \mathcal{A}_{n+1}(K)$ *for* $n \leq N$*, then there is an unramified prime* $\mathfrak{p}$ *satisfying both* $\cos\theta_{\mathfrak{p}} \in I$ *and*

$$\mathrm{N}\mathfrak{p} \ll ([K : \mathbb{Q}]^{[K:\mathbb{Q}]} N^N \mathfrak{q}(\pi)^{N^3})^{c_6 N^4 \log(3BN)}.$$

If $K = \mathbb{Q}$ *and* π *is associated to a non-CM newform with squarefree level, then this may be improved to*

$$\mathrm{N}\mathfrak{p} \ll (N\mathfrak{q}(\pi))^{c_6 N^5 \log(3BN)}.$$

Remarks. 1. Even if $\zeta_K(s)$ has a Landau-Siegel zero, we can still prove an effective bound for the least norm of a prime ideal in the Sato-Tate conjecture, but the bound will have a less desirable dependence on K. See the remark following the proof of Theorem 1.9.

2. When I is fixed and π varies, the bound in Theorem 1.9 has the shape $\mathrm{N}\mathfrak{p} \leq \mathfrak{q}(\pi)^A$ for some absolute constant A, and so is comparable to Linnik's theorem. However, if π is fixed and I is varying, the dependence is much worse. This comes partially from the constants in the zero-free region for $L(s, \mathrm{Sym}^n\pi, K)$ (see Lemma 3.3), where the n dependence in particular is of the form n^4. Without improving the quality of the dependence on n, it seems likely that only minor improvements can be made to Theorem 1.9.

3. Suppose that $\pi \in \mathcal{A}_2(K)$ is self-dual and genuine. It follows from work of Hoffstein and Ramakrishnan [22]; Goldfeld, Hoffstein, and Liemann [21] and Banks [3]; and Ramakrishnan and Wang [46] that none of $L(s, \pi, K)$, $L(s, \mathrm{Sym}^2\pi, K)$, and $L(s, \mathrm{Sym}^4\pi, K)$ has a Landau-Siegel zero (respectively). In fact, the proof of [22, Theorem B] also shows that if $L(s, \mathrm{Sym}^j\pi, K)$ is automorphic for $j \in \{n-2, n, n+2\}$, then $L(s, \mathrm{Sym}^n\pi, K)$ does not have a Landau-Siegel zero. From the known automorphy results mentioned earlier, it follows that if π corresponds with a non-CM Hilbert modular form, then $L(s, \mathrm{Sym}^n\pi, K)$ does not have a Landau-Siegel zero for $n = 1$, 2, and 4, and additionally $n = 3$, 5, and 6 if $K \cap \mathbb{Q}(e^{2\pi i/35}) = \mathbb{Q}$.

4. Following the ideas of Moreno [39, Theorem 4.2], we could prove a version of the zero repulsion phenomenon of Deuring and Heilbronn for $L(s, \pi \otimes \pi', K)$. Such a result would allow us to weaken the definition of I not admitting Landau-Siegel zeros. In particular, we would only need to require that for every $1 \leq n \leq N$ such that $L(s, \mathrm{Sym}^n\pi, K)$ has a Landau-Siegel zero, the coefficient b_n satisfies $b_n \leq b_0$. Since this does not completely eliminate the Landau-Siegel zero contribution, we do not carry out this computation.

5. If $K = \mathbb{Q}$, one may use Corollary 1.4 instead of Corollary 1.2 in the proof of Theorem 1.9. This would produce a bound on the least prime $p \equiv a \pmod{q}$ such that $\cos\theta_p \in I$.

Finally, we note that Corollary 1.4 can be used to study the distribution of values of automorphic L-functions in families of twists near the edge of the critical strip. For example, Lamzouri [31] extended the method of Granville and Soundararajan [18] for studying the distribution of $L(1, \chi_d, \mathbb{Q})$ (where χ_d is a primitive quadratic Dirichlet character to modulus d) to study the distribution of $L(1, \pi \otimes \chi_d, \mathbb{Q})$ where π is a cuspidal automorphic representation over $\mathbb{Q}$. Lamzouri's results rely on the existence of a log-free zero density estimate (which he called a "weak zero density estimate") averaged over twists by χ_d, which is known for some π by the aforementioned work of Kowalski and Michel [30]; Corollary 1.4 unconditionally provides such an estimate for all π. Corollary 1.4 is also used by Pasten [44, Section 17] to control the distribution of $\frac{L'}{L}(1, \pi \otimes \chi)$ in his proof that appropriate bounds for modular degrees imply Szpiro's conjecture in the context of modular elliptic curves over totally real fields other than $\mathbb{Q}$.

This paper is organized as follows. In Sections 2 and 3, we discuss the basic properties of automorphic L-functions that we will use in the proofs of the theorems; we also prove a few useful lemmas. In Section 4, we prove Theorems 1.1 and 1.3. In Section 5, we prove Theorems 1.6, 1.8, and 1.9. In Section 6, we prove Theorem 1.7.

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2. Rankin-Selberg L-functions

2.1. **Automorphic L-functions.** We follow the account of Rankin-Selberg L-functions given by Brumley [8, Section 1]. Let $K/\mathbb{Q}$ be a number field of absolute discriminant D_K. Let $\mathbb{A}_K$ denote the ring of adeles over K, and let $\mathcal{A}_d(K)$ be the set of cuspidal automorphic representations of $\mathrm{GL}_d(\mathbb{A}_K)$ with unitary central character.

Let $\pi \in \mathcal{A}_d(K)$. We have the factorization $\pi = \otimes_{\mathfrak{v}} \pi_{\mathfrak{v}}$ over the places of K. For each nonarchimedean $\mathfrak{p}$, we have the Euler factor

$$L_{\mathfrak{p}}(s, \pi, K) = \prod_{j=1}^{d} (1 - \alpha_\pi(j, \mathfrak{p}) \mathrm{N}\mathfrak{p}^{-s})^{-1}$$

associated with $\pi_{\mathfrak{p}}$. Let R_π be the set of prime ideals $\mathfrak{p}$ for which $\pi_{\mathfrak{p}}$ is ramified. We call $\alpha_\pi(j, \mathfrak{p})$ the local roots of $L(s, \pi, K)$ at $\mathfrak{p}$, and if $\mathfrak{p} \notin R_\pi$, then $\alpha_\pi(j, \mathfrak{p}) \neq 0$ for all $1 \leq j \leq d$. The representation π has an associated automorphic L-function whose Euler product and Dirichlet series are given by

$$L(s, \pi, K) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(s, \pi, K) = \sum_{\mathfrak{a}} \frac{\lambda_\pi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s},$$

where $\mathfrak{p}$ runs through the finite primes and $\mathfrak{a}$ runs through the non-zero integral ideals of K. This Euler product converges absolutely for $\mathrm{Re}(s) > 1$, which implies that $|\alpha_\pi(j, \mathfrak{p})| < \mathrm{N}\mathfrak{p}$. Luo, Rudnick, and Sarnak [33, Theorem 2] showed that if $\mathfrak{p} \notin R_\pi$, then

$$|\alpha_\pi(j, \mathfrak{p})| \leq \mathrm{N}\mathfrak{p}^{\frac{1}{2} - \frac{1}{d^2+1}}, \tag{2.1}$$

and Müller and Speh [43] proved that this holds for all primes $\mathfrak{p}$. The generalized Ramanujan conjecture (GRC) asserts a further improvement.

The generalized Ramanujan conjecture (GRC)**.** *Assume the above notation. For each prime* $\mathfrak{p} \notin R_\pi$*, we have* $|\alpha_\pi(j,\mathfrak{p})| = 1$*, and for each prime* $\mathfrak{p} \in R_\pi$*, we have* $|\alpha_\pi(j,\mathfrak{p})| \leq 1$*.*

Remark. It is expected that all automorphic L-functions $L(s,\pi,K)$ satisfy GRC. Indeed, it is already known for many of the most commonly used automorphic L-functions. Such L-functions include L-functions for $\pi \in \mathcal{A}_1(K)$ and, by Deligne [12], the L-function of a cuspidal normalized Hecke eigenform of positive even integer weight k on the congruence subgroup $\Gamma_0(N)$. More generally, Blasius [6] proved that Hilbert modular forms over totally real number fields with each weight both even and at least 2 satisfy GRC.

At each archimedean place $\mathfrak{v}$, we associate to $\pi_\mathfrak{v}$ a set of d complex numbers $\{\mu_\pi(j,\mathfrak{v})\}_{j=1}^d$, often called Langlands parameters, which are known to satisfy

$$\mathrm{Re}(\mu_\pi(j,\mathfrak{v})) > -\frac{1}{2} + \frac{1}{d^2+1},$$

again by [33, 43]. The local factor at $\mathfrak{v}$ is defined to be

$$L_\mathfrak{v}(s,\pi,K) = \prod_{j=1}^{d} \Gamma_{K_\mathfrak{v}}(s + \mu_\pi(j,\mathfrak{v})),$$

where $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$ and $\Gamma_\mathbb{C}(s) = \Gamma_\mathbb{R}(s)\Gamma_\mathbb{R}(s+1)$. Letting S_∞ denote the set of archimedean places, we define the gamma factor of $L(s,\pi,K)$ by

$$L_\infty(s,\pi,K) = \prod_{\mathfrak{v}\in S_\infty} L_\mathfrak{v}(s,\pi,K).$$

For notational convenience, we will define the complex numbers $\kappa_\pi(j)$ by

$$L_\infty(s,\pi,K) = \prod_{j=1}^{d[K:\mathbb{Q}]} \Gamma_\mathbb{R}(s + \kappa_\pi(j)). \tag{2.2}$$

Any automorphic L-function $L(s,\pi,K)$ admits a meromorphic continuation to $\mathbb{C}$ with poles possible only at $s=0$ and 1. Let $r(\pi)$ denote the order of the pole at $s=1$, and define the completed L-function

$$\Lambda(s,\pi,K) = (s(1-s))^{r(\pi)} q(\pi)^{s/2} L_\infty(s,\pi,K)L(s,\pi,K),$$

where $q(\pi)$ is the conductor of π. (Note that D_K^d divides $q(\pi)$.) It is well-known that $\Lambda(s,\pi,K)$ is an entire function of order 1 and that there exists a complex number $\varepsilon(\pi)$ of modulus 1 such that $\Lambda(s,\pi,K)$ satisfies the functional equation

$$\Lambda(s,\pi,K) = \varepsilon(\pi)\Lambda(1-s,\tilde{\pi},K),$$

where $\tilde{\pi}$ is the representation contragredient to π. For each $\mathfrak{p}$, we have that

$$\{\alpha_{\tilde{\pi}}(j,\mathfrak{p})\}_{j=1}^d = \{\overline{\alpha_\pi(k,\mathfrak{p})}\}_{k=1}^d.$$

Moreover,

$$L_\infty(s,\tilde{\pi},K) = L_\infty(s,\pi,K) \quad \text{and} \quad q(\tilde{\pi}) = q(\pi).$$

To maintain uniform estimates for the analytic quantities associated to $L(s,\pi,K)$, we define the **analytic conductor** of $L(s,\pi,K)$ by

$$\mathfrak{q}(s,\pi)=q(\pi)\prod_{j=1}^{d[K:\mathbb{Q}]}(|s+\kappa_\pi(j)|+3). \tag{2.3}$$

We will frequently make use of the quantity $\mathfrak{q}(0,\pi)$, which we denote by $\mathfrak{q}(\pi)$.

As in the introduction, define the von Mangoldt function $\Lambda_\pi(\mathfrak{a})$ by

$$-\frac{L'}{L}(s,\pi,K)=\sum_{\mathfrak{a}}\frac{\Lambda_\pi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s},$$

and let $\Lambda_K(\mathfrak{a})$ be that associated to the Dedekind zeta function $\zeta_K(s)$. We then have that $\Lambda_\pi(\mathfrak{a})$ is supported on powers of prime ideals, and

$$\Lambda_\pi(\mathfrak{p}^m)=\Lambda_K(\mathfrak{p}^m)\sum_{j=1}^{d}\alpha_\pi(j,\mathfrak{p})^m.$$

In particular, $\Lambda_\pi(\mathfrak{p})=\lambda_\pi(\mathfrak{p})\log\mathrm{N}\mathfrak{p}$. Using the bounds for $|\alpha_\pi(j,\mathfrak{p})|$ from [33, 43], we have that

$$|\Lambda_\pi(\mathfrak{a})|\leq d\Lambda_K(\mathfrak{a})\mathrm{N}\mathfrak{a}^{\frac{1}{2}-\frac{1}{d^2+1}} \tag{2.4}$$

for every ideal $\mathfrak{a}$, and under GRC, we have $|\Lambda_\pi(\mathfrak{a})|\leq d\Lambda_K(\mathfrak{a})$.

2.2. **Rankin-Selberg L-functions.** Consider two representations $\pi\in\mathcal{A}_d(K)$ and $\pi'\in\mathcal{A}_{d'}(K)$. We are interested in the Rankin-Selberg product $\pi\otimes\pi'$ of π and π', which, at primes $\mathfrak{p}\notin R_\pi\cap R_{\pi'}$, has a local factor given by

$$L_{\mathfrak{p}}(s,\pi\otimes\pi',K)=\prod_{j_1=1}^{d}\prod_{j_2=1}^{d'}(1-\alpha_\pi(j_1,\mathfrak{p})\alpha_{\pi'}(j_2,\mathfrak{p})\mathrm{N}\mathfrak{p}^{-s})^{-1}.$$

For $\mathfrak{p}\in R_\pi\cap R_{\pi'}$, we write the local roots as $\beta_{\pi\otimes\pi'}(j,\mathfrak{p})$ with $1\leq j\leq d'd$, and for each such $\mathfrak{p}$ we define

$$L_{\mathfrak{p}}(s,\pi\otimes\pi',K)=\prod_{j=1}^{d'd}(1-\beta_{\pi\otimes\pi'}(j,\mathfrak{p})\mathrm{N}\mathfrak{p}^{-s})^{-1}.$$

This gives rise to the Rankin-Selberg L-function $L(s,\pi\otimes\pi',K)$, whose Euler product and gamma factor are given by

$$L(s,\pi\otimes\pi',K)=\prod_{\mathfrak{p}}L_{\mathfrak{p}}(s,\pi\otimes\pi',K)$$

and

$$L_\infty(s,\pi\otimes\pi',K)=\prod_{\mathfrak{v}\in S_\infty}\prod_{j_1=1}^{d}\prod_{j_2=1}^{d'}\Gamma_{K_{\mathfrak{v}}}(s+\mu_{\pi\otimes\pi'}(j_1,j_2,\mathfrak{v}))=\prod_{j=1}^{d'd[K:\mathbb{Q}]}\Gamma_{\mathbb{R}}(s+\kappa_{\pi\otimes\pi'}(j)),$$

where

$$\mathrm{Re}(\kappa_{\pi\otimes\pi'}(j))\geq-1+\frac{1}{d^2+1}+\frac{1}{(d')^2+1} \tag{2.5}$$

for all $1 \leq j \leq d'd[K:\mathbb{Q}]$ and

$$|\alpha_{\pi\otimes\pi'}(j_1,j_2,\mathfrak{p})| \leq \mathrm{N}\mathfrak{p}^{1-\frac{1}{d^2+1}-\frac{1}{(d')^2+1}} \tag{2.6}$$

for all $\mathfrak{p}$. (See [8, Section 1] for further discussion regarding these bounds at ramified places.)

By [8, Equation 8], we have

$$\mathfrak{q}(s,\pi\otimes\pi') \leq \mathfrak{q}(\pi)^{d'}\mathfrak{q}(\pi')^{d}(|s|+3)^{d'd[K:\mathbb{Q}]}. \tag{2.7}$$

Moreover, if $K=\mathbb{Q}$, χ a primitive Dirichlet character modulo q, and $\gcd(q,q(\pi)q(\pi'))=1$, then

$$\mathfrak{q}(s,(\pi\otimes\chi)\otimes\pi') \leq \mathfrak{q}(\pi)^{d'}\mathfrak{q}(\pi')^{d}q^{d'd}(|s|+3)^{d'd}. \tag{2.8}$$

Finally, we note that if $\pi'\cong\tilde{\pi}$, then the order $r(\pi\otimes\pi')$ of the pole at $s=1$ is 1.

3. Preliminaries

3.1. **Zero-free regions.** We require regions of the critical strip which contain few nontrivial zeros of $L(s,\pi\otimes\pi',K)$. In the spirit of Mertens's variant of the proof of the zero-free region for $\zeta(s)$ originally due to Hadamard and de la Vallée Poussin, one typically finds a suitable Dirichlet series $D(s)$ with nonnegative Dirichlet coefficients such that $L(s,\pi\otimes\pi',K)$ divides $D(s)$ with a multiplicity m that is (strictly) larger than the order of the pole of $D(s)$ at $s=1$. A verification of the holomorphy of $D(s)/L(s,\pi\otimes\pi',K)^m$ in some interval $(t,1)$ for a fixed $0<t<1$ then yields the desired zero-free region.

In practice, one can find a region in the critical strip for which $L(s,\pi\otimes\pi',K)$ is nonzero with the possible exception of a simple real zero near $s=1$ whenever at least one of π and π' is self-dual. (See [17, 39] for further discussion.) However, since it is our purpose only to count zeros, it suffices to establish regions of the critical strip which contain an *absolutely bounded* number of zeros. Such a result comes readily for $L(s,\pi\otimes\pi',K)$, even if neither π nor π' is self-dual, via the following lemma, which descends from the appendix to [21].

Lemma 3.1. *Let Π be an isobaric representation of $\mathrm{GL}_d(\mathbb{A}_K)$. If $r(\Pi\otimes\tilde{\Pi})\geq 1$ is the order of the pole of $L(s,\Pi\otimes\tilde{\Pi},K)$ at $s=1$, then there exists an absolute constant [2] $c_7>0$ such that $\Lambda(\sigma,\Pi\otimes\tilde{\Pi},K)$ has at most $r(\Pi\otimes\tilde{\Pi})$ real zeros in the region*

$$\sigma \geq 1-\frac{c_7}{(r(\Pi\otimes\tilde{\Pi})+1)\log\mathfrak{q}(\Pi\otimes\tilde{\Pi})}.$$

Proof. This is proved by Hoffstein and Ramakrishnan [22, Lemma c], but they only showed that c_7 depends on at most $r(\Pi\otimes\tilde{\Pi})$ and d. The dependence can be made explicit by proceeding as in [25, Lemma 5.9], though the degree dependence in [25, Lemma 5.9] can be removed in our case because, as shown in [22, Lemma a], all Dirichlet coefficients of $L(s,\Pi\otimes\tilde{\Pi})$ are real and nonnegative. □

Let $T\geq 1$. For future convenience, we define

$$\mathcal{Q}=\begin{cases}\max\{\mathfrak{q}(\pi),D_K[K:\mathbb{Q}]^{[K:\mathbb{Q}]}\} & \text{if } d'=1 \text{ and } \pi' \text{ is trivial,}\\ \max\{\mathfrak{q}(\pi)\mathfrak{q}(\pi'),D_K[K:\mathbb{Q}]^{[K:\mathbb{Q}]}\} & \text{otherwise}\end{cases}$$

[2] We denote by $c_1,c_2,\ldots$ a sequence of constants, each of which is absolute, positive, and effectively computable. We do not recall this convention in future statements, as we find it to be notationally cumbersome.

and

$$\mathcal{L} = \mathcal{L}(T, \pi \otimes \pi', K) = d'd \log(\mathcal{Q}T^{[K:\mathbb{Q}]}). \tag{3.1}$$

Note that from the definition of $\mathcal{L}$, we always have $\mathcal{L} \gg d'd[K:\mathbb{Q}]$.

Lemma 3.2. *Let $T \geq 1$. If c_8 is sufficiently small, then the region*

$$\{s = \sigma + it \colon 1 - c_8\mathcal{L}^{-1} \leq \sigma < 1, |t| \leq T\}$$

contains at most four zeros of $L(s, \pi \otimes \pi', K)$.

Proof. This follows from applying Lemma 3.1 to the isobaric representation $\Pi = (\pi \otimes |\det|^{it/2}) \boxplus (\tilde{\pi}' \otimes |\det|^{-it/2})$; see also [39, Section 3]. □

When $d' = 1$ and π' is trivial, we can obtain a tighter result regarding the location of the zeros at the cost of a slightly worse dependence on d.

Lemma 3.3. *Let $T \geq 1$. The region*

$$\{s = \sigma + it \colon 1 - c_8(d^3\mathcal{L})^{-1} \leq \sigma < 1, |t| \leq T\}$$

contains no zeros of $L(s, \pi, K)$ except possibly for one simple real zero β_1, in which case π is self-dual. We call such a zero β_1 a Landau-Siegel zero.

Proof. The proof is the same as [25, Lemma 5.9]. □

3.2. **Useful bounds.** Let $S = S(\pi \otimes \pi') = R_\pi \cap R_{\pi'}$, and define the partial L-function

$$L^S(s, \pi \otimes \pi', K) = \prod_{\mathfrak{p} \notin S} L_{\mathfrak{p}}(s, \pi \otimes \pi', K).$$

We write $(\mathfrak{a}, S) = 1$ to say that the prime factors of $\mathfrak{a}$ do not lie in S. If $(\mathfrak{a}, S) = 1$, then $\lambda_{\pi\otimes\pi'}(\mathfrak{a}) = \lambda_\pi(\mathfrak{a})\lambda_{\pi'}(\mathfrak{a})$. Define

$$N^S_{\pi\otimes\pi'}(\sigma, T) = \#\{\rho = \beta + i\gamma \colon L^S(\pi, \pi \otimes \pi', K) = 0, \beta \geq \sigma, |\gamma| \leq T\}.$$

If $1 - (d^2+1)^{-1} - ((d')^2+1)^{-1} \leq \sigma \leq 1$, it is clear that $N^S_{\pi\otimes\pi'}(\sigma, T) = N_{\pi\otimes\pi'}(\sigma, T)$. We will prove Theorem 1.1 and its variants for $L^S(s, \pi \otimes \pi', K)$ and then deduce the results for $L(s, \pi \otimes \pi', K)$. Note that the zero-free regions in Section 3.1 also apply to $L^S(s, \pi \otimes \pi', K)$.

Lemma 3.4. *Let $T \geq 0$, and let $\tau \in \mathbb{R}$ satisfy $|\tau| \leq T$.*

(1) Uniformly on the disk $|s - (1 + i\tau)| \leq 1/4$, we have that

$$\frac{(L^S)'}{L^S}(s, \pi \otimes \pi', K) + \frac{r(\pi \otimes \pi')}{s} + \frac{r(\pi \otimes \pi')}{s-1} - \sum_{|\rho - (1+i\tau)| \leq 1/2} \frac{1}{s - \rho} \ll \mathcal{L},$$

where the sum runs over zeros ρ of $L^S(s, \pi \otimes \pi', K)$.

(2) For $1 \geq \eta \gg \mathcal{L}^{-1}$, we have that

$$\sum_{|\rho - (1+i\tau)| \leq \eta} 1 \ll \eta\mathcal{L},$$

where the sum runs over zeros ρ of $L^S(s, \pi \otimes \pi', K)$.

Proof. Part 1 is a slight variation of [25, Equation 5.28]; see [1, Lemma 2.4], for example. Part 2 follows from combining [25, Theorem 5.6] and [25, Proposition 5.8]. □

Lemma 3.5. *If $\eta > 0$ and $y \geq 2$, then*

(1) $\displaystyle\sum_{(\mathfrak{a},S)=1}\frac{|\Lambda_{\pi\otimes\pi'}(\mathfrak{a})|}{\mathrm{N}\mathfrak{a}^{1+\eta}}\ll\frac{1}{\eta}+d'd\log(\mathfrak{q}(\pi)\mathfrak{q}(\pi'))$.

(2) $\displaystyle\sum_{\substack{\mathrm{N}\mathfrak{a}\leq y\\(\mathfrak{a},S)=1}}\frac{|\Lambda_{\pi\otimes\pi'}(\mathfrak{a})|}{\mathrm{N}\mathfrak{a}}\ll\log y+d'd\log(\mathfrak{q}(\pi)\mathfrak{q}(\pi'))$.

For convenience, we write $(\mathfrak{a},S)=1$ if the ideal $\mathfrak{a}$ has no prime factors in S.

Proof. By the Cauchy-Schwarz inequality, we have

$$\sum_{(\mathfrak{a},S)=1}\frac{|\Lambda_{\pi\otimes\pi'}(\mathfrak{a})|}{(\mathrm{N}\mathfrak{a})^{1+\eta}}\ll\Big(-\frac{L'}{L}(1+\eta,\pi\otimes\tilde{\pi},K)\Big)^{1/2}\Big(-\frac{L'}{L}(1+\eta,\pi'\otimes\tilde{\pi}',K)\Big)^{1/2}.$$

We first estimate $-\frac{L'}{L}(1+\eta,\pi\otimes\tilde{\pi},K)$, which is a positive quantity because $\eta>0$ and the Dirichlet coefficients of $-\frac{L'}{L}(s,\pi\otimes\tilde{\pi},K)$ are real and nonnegative. By [25, Theorem 5.6] and [25, Proposition 5.7], we have

$$-\mathrm{Re}\Big(\frac{L'}{L}(1+\eta,\pi\otimes\tilde{\pi},K)\Big)=\frac{1}{2}\log q(\pi\otimes\tilde{\pi})+\mathrm{Re}\Big(\frac{L'_\infty}{L_\infty}(1+\eta,\pi\otimes\tilde{\pi},K)\Big)+\frac{1}{1+\eta}+\frac{1}{\eta}-\sum_{\rho\neq0,1}\mathrm{Re}\Big(\frac{1}{1+\eta-\rho}\Big),$$

where $\rho=\beta+i\gamma$ runs through the zeros of $\Lambda(s,\pi\otimes\pi',K)$. Since $0\leq\beta<1$, we have

$$\mathrm{Re}\Big(\frac{1}{1+\eta-\rho}\Big)=\frac{1+\eta+\beta}{(1+\eta+\beta)^2+\gamma^2}>0,$$

the contribution from sum over zeros is negative, so we can discard it. Thus

$$-\mathrm{Re}\Big(\frac{L'}{L}(1+\eta,\pi\otimes\tilde{\pi},K)\Big)\leq\frac{1}{2}\log q(\pi\otimes\tilde{\pi})+\mathrm{Re}\Big(\frac{L'_\infty}{L_\infty}(1+\eta,\pi\otimes\tilde{\pi},K)\Big)+\frac{1}{1+\eta}+\frac{1}{\eta}.$$

By the proof of [25, Proposition 5.7, part 2], we have

$$\mathrm{Re}\Big(\frac{L'_\infty}{L_\infty}(1+\eta,\pi\otimes\tilde{\pi},K)\Big)=-\sum_{|s+\kappa_{\pi\otimes\tilde{\pi}}(j)|<1}\mathrm{Re}\Big(\frac{1}{1+\eta+\kappa_{\pi\otimes\tilde{\pi}}(j)}\Big)+O(\log\mathfrak{q}(\pi\otimes\tilde{\pi})).$$

Since $\mathrm{Re}(\kappa_{\pi\otimes\pi'}(j))>-1$ for all $1\leq j\leq d'd[K:\mathbb{Q}]$, it follows that $\mathrm{Re}(\frac{1}{1+\eta+\kappa_{\pi\otimes\tilde{\pi}}(j)})>0$. Therefore, by positivity and (2.7),

$$-\frac{L'}{L}(1+\eta,\pi\otimes\tilde{\pi},K)\ll\frac{1}{\eta}+\log(\mathfrak{q}(\pi\otimes\tilde{\pi}))\ll\frac{1}{\eta}+d\log\mathfrak{q}(\pi).$$

Since the analogue holds for π', part 1 follows. Part 2 follows by choosing $\eta=(\log y)^{-1}$. □

We conclude this section with bounds on the mean values of Dirichlet polynomials.

Lemma 3.6. *Let $T\geq1$ and $u>y\gg(\mathcal{Q}T^{[K:\mathbb{Q}]})^{c_9}$, where c_9 is sufficiently large. Define*

$$S_{y,u}(\tau,\pi\otimes\pi'):=\sum_{y<\mathrm{N}\mathfrak{p}\leq u}\frac{\Lambda_{\pi\otimes\pi'}(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^{1+i\tau}}.$$

1. If $L(s,\pi',K)$ satisfies GRC, then

$$\int_{-T}^{T}|S_{y,u}(\tau,\pi\otimes\pi')|^2d\tau\ll\frac{(d')^2(\log u)(\log u+d^2\log\mathfrak{q}(\pi))}{\log y}.$$

2. If $K=\mathbb{Q}$ and $L(s,\pi,\mathbb{Q})$ satisfies GRC, then

$$\sum_{\substack{q\leq T^2\\ \gcd(q,q(\pi)q(\pi'))=1}}\log\frac{T^2}{q}\sideset{}{^\star}\sum_{\chi \bmod q}\int_{-T}^{T}|S_{y,u}(\tau,(\pi\otimes\chi)\otimes\pi')|^2d\tau\ll(d')^2(\log u)(\log u+d^2\log\mathfrak{q}(\pi)).$$

Proof. 1. Let $b(\mathfrak{p})$ be a complex-valued function supported on the prime ideals of K such that $\sum_{\mathfrak{p}}|b(\mathfrak{p})|^2\mathrm{N}\mathfrak{p}<\infty$ and $b(\mathfrak{p})=0$ whenever $\mathrm{N}\mathfrak{p}\leq y$. With our choice of T and y, it follows from [52, Corollary 3.8] that

$$\int_{-T}^{T}\Big|\sum_{\mathfrak{p}}b(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-i\tau}\Big|^2d\tau\ll\frac{1}{\log y}\sum_{\mathfrak{p}}|b(\mathfrak{p})|^2\mathrm{N}\mathfrak{p},\tag{3.2}$$

If we define $b(\mathfrak{p})$ by

$$b(\mathfrak{p})=\begin{cases}\dfrac{\Lambda_{\pi\otimes\pi'}(\mathfrak{p})}{\mathrm{N}\mathfrak{p}} & \text{if } y<\mathrm{N}\mathfrak{p}\leq u,\\ 0 & \text{otherwise}\end{cases}$$

and recall the definition of $S_{y,u}(\tau,\pi\otimes\pi')$, then an application of (3.2) yields the bound

$$\int_{-T}^{T}|S_{y,u}(\tau,\pi\otimes\pi')|^2d\tau\ll\frac{1}{\log y}\sum_{y<\mathrm{N}\mathfrak{p}\leq u}\frac{|\Lambda_{\pi\otimes\pi'}(\mathfrak{p})|^2}{\mathrm{N}\mathfrak{p}}.$$

Since y is greater than the norm of any ramified prime, it follows from our assumption of GRC for $L(s,\pi',K)$ that

$$\sum_{y<\mathrm{N}\mathfrak{p}\leq u}\frac{|\Lambda_{\pi\otimes\pi'}(\mathfrak{p})|^2}{\mathrm{N}\mathfrak{p}}=\sum_{y<\mathrm{N}\mathfrak{p}\leq u}\frac{|\lambda_{\pi}(\mathfrak{p})|^2|\lambda_{\pi'}(\mathfrak{p})|^2(\log\mathrm{N}\mathfrak{p})^2}{\mathrm{N}\mathfrak{p}}\leq(d')^2\sum_{y<\mathrm{N}\mathfrak{p}\leq u}\frac{|\lambda_{\pi}(\mathfrak{p})|^2(\log\mathrm{N}\mathfrak{p})^2}{\mathrm{N}\mathfrak{p}}.$$

Since all prime ideals $\mathfrak{p}$ in the sum are unramified, we have that $|\lambda_{\pi}(\mathfrak{p})|^2\log\mathrm{N}\mathfrak{p}=|\Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{p})|$. The claimed result now follows by partial summation using Lemma 3.5.

2. Let $K=\mathbb{Q}$. Suppose that $a(p)$ is a function on primes such that $a(p)=0$ if $p\leq Q$ and $\sum_p|a(p)|^2p<\infty$. By [15, Theorem 4], we have that for $T\geq 1$,

$$\sum_{q\leq Q}\log\frac{Q}{q}\sideset{}{^\star}\sum_{\chi \bmod q}\int_{-T}^{T}\Big|\sum_{p}a(p)\chi(p)p^{-it}\Big|^2dt\ll\sum_{p}(Q^2T+p)|a(p)|^2.\tag{3.3}$$

Let $Q=T^2$ and

$$a(p)=\begin{cases}\dfrac{\Lambda_{\pi\otimes\pi'}(p)\chi(p)}{p} & \text{if } y<p\leq u,\\ 0 & \text{otherwise.}\end{cases}$$

Note that if $\gcd(q,q(\pi)q(\pi'))=1$, then $\Lambda_{\pi\otimes\pi'}(p)\chi(p)=\Lambda_{(\pi\otimes\chi)\otimes\pi'}(p)$; moreover, our choice of y implies that $a(p)=0$ at every ramified prime p dividing $q((\pi\otimes\chi)\otimes\pi')$. Choosing $c_9>6$, our hypotheses imply that $T^5 \ll p$ for every $p \in (u, y]$. Thus

$$\begin{aligned}&\sum_{\substack{q \le T^2 \\ \gcd(q, q(\pi)q(\pi'))=1}} \log \frac{T^2}{q} \sideset{}{^\star}\sum_{\chi \bmod q} \int_{-T}^{T} |S_{y,u}(\tau, (\pi\otimes\chi)\otimes\pi')|^2 d\tau \\ &\le \sum_{q \le T^2} \log \frac{T^2}{q} \sideset{}{^\star}\sum_{\chi \bmod q} \int_{-T}^{T} \Big| \sum_{y<p\le u} \frac{\Lambda_{\pi\otimes\pi'}(p)\chi(p)}{p^{1+i\tau}} \Big|^2 d\tau \\ &\ll \sum_{y<p\le u} (T^5+p) \frac{|\Lambda_{\pi\otimes\pi'}(p)|^2}{p^2} \le \sum_{y<p\le u} \frac{|\Lambda_{\pi\otimes\pi'}(p)|^2}{p}.\end{aligned}$$

This is bounded using GRC just as in the proof of Part 1. □

4. The zero density estimate

In this section, we prove Theorem 1.1 by generalizing Gallagher's [15] and Weiss's [52] treatment of Turán's method for detecting zeros of L-functions, obtaining a result that is uniform in K, π, and π'. The key result is the following technical proposition, whose proof we defer to the end of the section. Recall from Lemma 3.2 that $\mathcal{L} = d'd \log(QT^{[K:\mathbb{Q}]})$.

Proposition 4.1. *Recall the notation and hypotheses of Theorem 1.1. Let $y = e^{c_9\mathcal{L}}$ with c_9 sufficiently large. Suppose that η satisfies $\mathcal{L}^{-1} \ll \eta \le 1/55$. Let*

$$S_{y,u}(\tau, \pi\otimes\pi') := \sum_{y<\mathrm{N}\mathfrak{p}\le u} \frac{\Lambda_{\pi\otimes\pi'}(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^{1+i\tau}}.$$

If $L^S(s,\pi\otimes\pi')$ has a zero ρ_0 satisfying $|\rho_0 - (1+i\tau)| \le \eta$ and $\mathrm{Re}(\rho_0) \le 1 - c_8/\mathcal{L}$, then for sufficiently large c_{10} and c_{11}, we have that

$$\frac{y^{c_{10}\eta}}{(\log y)^3} \int_y^{y^{c_{11}}} |S_{y,u}(\tau,\pi\otimes\pi')|^2 \frac{du}{u} \gg 1.$$

We first deduce Theorem 1.1 from Proposition 4.1. The proof of Proposition 4.1 relies on certain upper and lower bounds on the derivatives of $\frac{(L^S)'}{L^S}(s,\pi\otimes\pi')$, which are proven and assembled subsequently.

4.1. Proof of Theorems 1.1 and 1.3.

By [25, Theorem 5.8],

$$N_{\pi\otimes\pi'}(0,T) = \frac{T}{\pi}\log\Big(q(\pi\otimes\pi')\Big(\frac{T}{2\pi e}\Big)^{d'd[K:\mathbb{Q}]}\Big) + O(\log \mathfrak{q}(iT,\pi\otimes\pi')) \tag{4.1}$$

for all $T \ge 1$; by a slight variation of the proof, the same bound holds for $N^S_{\pi\otimes\pi'}(0,T)$. Thus it suffices to prove Theorems 1.1 and 1.3 for $1-\sigma$ sufficiently small. Since the left side of Theorem 1.1 is a decreasing function of σ and the right side of Theorem 1.1 is essentially constant for $1-\sigma \ll \mathcal{L}^{-1}$, it suffices to prove the theorem for $1-\sigma \gg \mathcal{L}^{-1}$. Therefore, we may assume that $c_{12} \le \sigma \le 1 - c_8\mathcal{L}^{-1}$, where $\frac{1}{2} < c_{12} < 1$. Since c_8 and c_{12} are absolute, we may take c_8 sufficiently close to 0 and c_{12} sufficiently close to 1 such that we may take $\eta = \sqrt{2}(1-\sigma)$ in Proposition 4.1.

Suppose that $T \ge 1$ and $\rho = \beta + i\gamma$ satisfies $|\gamma| \le T$ and $\sigma \le \beta \le 1 - c_8\mathcal{L}^{-1}$. For $\tau \in \mathbb{R}$, let

$$\psi_\rho(\tau) = \begin{cases} 1 & \text{if } |\gamma-\tau| \le 1-\sigma \text{ and } |\tau| \le T, \\ 0 & \text{otherwise.}\end{cases}$$

Since $\eta = \sqrt{2}(1-\sigma)$, we have that $\sqrt{2}\int_{-T}^{T}\psi_\rho(\tau)d\tau \geq \eta$ for each ρ, while $\sum_\rho \psi_\rho(\tau) \ll \eta\mathcal{L}$ by Lemma 3.4. Thus

$$N_{\pi\otimes\pi'}^{S}(\sigma,T) \ll \int_{-T}^{T}\sum_{\rho}\eta^{-1}\psi_\rho(\tau)d\tau.$$

Since $\psi_\rho(\tau)\neq 0$ implies that $|\rho-(1+i\tau)|\leq\eta$, we have by Proposition 4.1 and the bound $1\ll\eta\mathcal{L}$ that

$$\begin{aligned}\int_{-T}^{T}\sum_{\rho}\eta^{-1}\psi_\rho(\tau)d\tau &\ll \int_{-T}^{T}\eta^{-1}\Big(\sum_{\rho}\psi_\rho(\tau)\Big)\frac{y^{c_{10}\eta}}{(\log y)^3}\Big(\int_{y}^{y^{c_{11}}}|S_{y,u}(\tau,\pi\otimes\pi')|^2\frac{du}{u}\Big)d\tau\\ &\ll \mathcal{L}\frac{y^{c_{10}\eta}}{(\log y)^3}\int_{y}^{y^{c_{11}}}\Big(\int_{-T}^{T}|S_{y,u}(\tau,\pi\otimes\pi')|^2 d\tau\Big)\frac{du}{u}.\end{aligned}$$

If π' satisfies GRC, then it follows from Part 1 of Lemma 3.6, the definition of $S_{y,u}(\tau,\pi\otimes\pi')$, and the fact that $y=e^{c_9\mathcal{L}}$ (with c_9 sufficiently large) that

$$N_{\pi\otimes\pi'}^{S}(\sigma,T) \ll (d')^2\mathcal{L}\frac{y^{c_{10}\eta}}{(\log y)^4}\int_{y}^{y^{c_{11}}}\frac{(\log u)(\log u + d^2\log\mathfrak{q}(\pi))}{u}du \ll (d')^2 y^{c_{10}\eta}.$$

Since $\eta=\sqrt{2}(1-\sigma)$ and $y=e^{c_9\mathcal{L}}$, we have

$$N_{\pi\otimes\pi'}^{S}(\sigma,T) \ll (d')^2 y^{c_{10}\sqrt{2}(1-\sigma)} \ll (d')^2(\mathcal{Q}T^{[K:\mathbb{Q}]})^{\sqrt{2}c_9c_{10}d'd(1-\sigma)}.$$

We let $c_1 = 24\sqrt{2}c_9c_{10}$, so that

$$N_{\pi\otimes\pi'}^{S}(\sigma,T) \ll (d')^2(\mathcal{Q}T^{[K:\mathbb{Q}]})^{c_1d'd(1-\sigma)/24}.$$

To conclude the proof of Theorem 1.1, we make a few small observations. First, note that $D_K^{d+d'}$ divides $q(\pi)q(\pi')$. Thus if we replace the condition $T\geq 1$ with $T\geq[K:\mathbb{Q}]$, then $\mathcal{Q}T^{[K:\mathbb{Q}]} \ll (\mathfrak{q}(\pi)\mathfrak{q}(\pi')T^{[K:\mathbb{Q}]})^2$, giving the slightly tidier bound

$$N_{\pi\otimes\pi'}^{S}(\sigma,T) \ll (d')^2(\mathfrak{q}(\pi)\mathfrak{q}(\pi')T^{[K:\mathbb{Q}]})^{c_1d'd(1-\sigma)/12}.$$

Second, our method only detects zeros of $\Lambda(s,\pi\otimes\pi',K)$ with the Euler factors at the ramified primes removed, so we must account for the $O(d'd[K:\mathbb{Q}])$ "trivial zeros" which arise as poles of $L_\infty(s,\pi\otimes\pi',K)$ along with the $O(\log(q(\pi)q(\pi')))$ zeros that arise from the removed Euler factors. However, the real parts of these zeros are no larger than $1-(d^2+1)^{-1}-((d')^2+1)^{-1}$ by (2.5) and (2.6). Thus for $\sigma\in[1-(d^2+1)^{-1}-((d')^2+1)^{-1},1)$, we have $N_{\pi\otimes\pi'}(\sigma,T)=N_{\pi\otimes\pi'}^{S}(\sigma,T)$. If $1/2\leq\sigma<1-(d^2+1)^{-1}-((d')^2+1)^{-1}$, then the bound for $N_{\pi\otimes\pi'}(\sigma,T)$ in Theorem 1.1 is trivial when compared to (4.1). Finally, we increase the implied constant to account for the four possible zeros in Lemma 3.2, and Theorem 1.1 follows. To obtain Corollary 1.2, note that if π' is the trivial representation of $\mathrm{GL}_1(\mathbb{A}_K)$, then $\mathfrak{q}(\pi')\leq D_K(T+4)^{[K:\mathbb{Q}]}$, so we find that

$$N_\pi(\sigma,T) \ll (\mathfrak{q}(\pi)D_K(T+4)^{[K:\mathbb{Q}]}T^{[K:\mathbb{Q}]})^{c_1d(1-\sigma)/4} \ll (\mathfrak{q}(\pi)T^{[K:\mathbb{Q}]})^{c_1d(1-\sigma)/2}.$$

Theorem 1.3 and Corollary 1.4 are proven similarly using (2.8), but we use part 2 of Lemma 4.2 instead of part 1.

4.2. **Bounds on derivatives.** We begin the proof of Proposition 4.1 by introducing notation which we will use throughout this section and the next. First, let $r = r(\pi \otimes \pi')$ be the order of the possible pole of $L^S(s, \pi \otimes \pi', K)$ at $s = 1$. We suppose that $L^S(s, \pi \otimes \pi', K)$ has a zero ρ_0 with $\mathrm{Re}(\rho_0) \leq 1 - c_8/\mathcal{L}$ satisfying

$$|\rho_0 - (1 + i\tau)| \leq \eta,$$

and we set

$$F(s) = \frac{(L^S)'}{L^S}(s, \pi \otimes \pi', K).$$

Suppose that $|\tau| \leq T$, where $T \geq 1$, as in the statement of Proposition 4.1. On the disk $|s - (1 + i\tau)| < 1/4$, by part 1 of Lemma 3.4, we have

$$F(s) + \frac{r}{s} + \frac{r}{s-1} = \sum_{|\rho - (1+i\tau)| \leq 1/2} \frac{1}{s - \rho} + G(s),$$

where $G(s)$ is analytic and $|G(s)| \ll \mathcal{L}$. Setting $\xi = 1 + \eta + i\tau$, we have

$$\frac{(-1)^k}{k!} \frac{d^k F}{ds^k}(\xi) + r(\xi - 1)^{-(k+1)} = \sum_{|\rho - (1+i\tau)| \leq 1/2} (\xi - \rho)^{-(k+1)} + O(8^k \mathcal{L}), \tag{4.2}$$

where the error term absorbs the contribution from integrating $G(s)$ over a circle of radius 1/8 centered at ξ and the term coming from differentiating $\frac{r}{s}$. We begin by obtaining a lower bound on the derivatives of $F(s)$.

Lemma 4.2. *Assume the notation above. For any $M \gg \eta\mathcal{L}$, there is some $k \in [M, 2M]$ such that*

$$\frac{\eta^{k+1}}{k!} \left| \frac{d^k F}{ds^k}(\xi) \right| \geq \frac{1}{2}(100)^{-(k+1)},$$

where $\xi = 1 + \eta + i\tau$.

We prove Lemma 4.2 by using a version of Turán's [51] power-sum estimate.

Lemma 4.3 (Turán)**.** *Let $z_1, \ldots, z_m \in \mathbb{C}$. If $M \geq m$, then there exists $k \in \mathbb{Z} \cap [M, 2M]$ such that $|z_1^k + \cdots + z_m^k| \geq (\frac{1}{50}|z_1|)^k$.*

Let $M = 300\eta \log y$. By our choices of η, $\mathcal{L}$, y, M, and k, we have the useful relationship

$$1 \ll \eta\mathcal{L} \asymp \eta \log y \asymp M \asymp k. \tag{4.3}$$

Proof of Lemma 4.2. We begin by considering the contribution to (4.2) from those zeros ρ satisfying $200\eta < |\rho - (1 + i\tau)| \leq 1/2$. In particular, by decomposing the sum dyadically and applying part 2 of Lemma 3.4, we find that

$$\sum_{200\eta < |\rho - (1+i\tau)| \leq 1/2} |\rho - \xi|^{-(k+1)} \ll \sum_{j=0}^{\infty} (2^j 200\eta)^{-(k+1)} 2^{j+1} r\mathcal{L} \ll (200\eta)^{-k}\mathcal{L},$$

This shows that it suffices to consider the zeros ρ for which $|\rho - (1 + i\tau)| \leq 200\eta$.

Since $0 < \eta \leq 1/55$, we have

$$\frac{1}{k!} \frac{d^k F}{ds^k}(\xi) + r(\xi - 1)^{-(k+1)} \geq \left| \sum_{|\rho - (1+i\tau)| \leq 200\eta} (\xi - \rho)^{-(k+1)} \right| - O((200\eta)^{-k}\mathcal{L}). \tag{4.4}$$

By Lemma 3.4 (part 2), the sum over zeros has $\ll \eta\mathcal{L}$ terms. Since $M \asymp \eta\mathcal{L}$, Lemma 4.3 tells us that for some $k \in [M, 2M]$, the sum over zeros on the right side of (4.4) is bounded below by $(50|\xi - \rho_0|)^{-(k+1)}$, where ρ_0 is the nontrivial zero which is being detected.

Since $|\xi - \rho_0| \leq 2\eta$, the right side of the above inequality is bounded below by

$$(100\eta)^{-(k+1)}(1 - O(2^{-k}\eta\mathcal{L})).$$

Since $k \geq M \gg \eta\mathcal{L}$ and $\mathcal{L}^{-1} \ll \eta \ll 1$, there is a constant $0 < \theta < 1$ so that

$$2^{-k}\eta\mathcal{L} = O(\theta^{\eta\mathcal{L}}\eta\mathcal{L}) \leq 1/4.$$

Therefore, for some $k \in [M, 2M]$ with $M \gg \eta\mathcal{L}$, we have

$$\frac{\eta^{k+1}}{k!}\Big|\frac{d^k F}{ds^k}(\xi)\Big| + r\eta^{k+1}|(\xi - 1)^{-(k+1)}| \geq \frac{3}{4}(100)^{-(k+1)}.$$

During the proof of Theorem 4.2 in [52], Weiss proves that

$$r\eta^{k+1}|(\xi - 1)^{-(k+1)}| \leq \frac{1}{4}(100)^{-(k+1)}.$$

The desired result now follows, that $\frac{\eta^{k+1}}{k!}|\frac{d^k F}{ds^k}(\xi)| \geq \frac{1}{2}(100)^{-(k+1)}$. □

We now turn to obtaining an upper bound on the derivatives of $F(s)$, for which we have the following.

Lemma 4.4. *Assume the notation preceeding Lemma 4.2. Let $M = 300\eta \log y$, and let k be determined by Lemma 4.2. Then*

$$\frac{\eta^{k+1}}{k!}\Big|\frac{d^k F}{ds^k}(\xi)\Big| \leq \eta^2 \int_y^{y^{c_{11}}} |S_{y,u}(\tau, \pi \otimes \pi')|\frac{du}{u} + \frac{1}{4}(100)^{-(k+1)},$$

where $S_{y,u}(\tau, \pi \otimes \pi')$ is as in Proposition 4.1.

Proof. Let $M = 300\eta \log y$ and $y = e^{c_9\mathcal{L}}$ for some c_9, which we will take to be sufficiently large. For $u > 0$, define

$$j_k(u) = \frac{u^k e^{-u}}{k!},$$

which satisfies

$$j_k(u) \leq \begin{cases} (100)^{-k} & \text{if } u \leq k/300, \\ (110)^{-k}e^{-u/2} & \text{if } u \geq 20k. \end{cases}$$

Letting $c_{11} \geq 12000$ be sufficiently large, we thus have

$$j_k(\eta \log(\mathrm{N}\mathfrak{a})) \leq \begin{cases} (110)^{-k} & \text{if } \mathrm{N}\mathfrak{a} \leq y, \\ (100)^{-k}(\mathrm{N}\mathfrak{a})^{-\eta/2} & \text{if } \mathrm{N}\mathfrak{a} \geq y^{c_{11}}. \end{cases} \tag{4.5}$$

Differentiating the Dirichlet series for $F(s)$ directly, we obtain

$$\frac{(-1)^{k+1}\eta^{k+1}}{k!}\frac{d^k F}{ds^k}(\xi) = \eta \sum_{(\mathfrak{a},S)=1} \frac{\Lambda_{\pi\otimes\pi'}(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^{1+i\tau}} j_k(\eta \log(\mathrm{N}\mathfrak{a}))$$

Splitting the above sum $\sum$ in concert with the inequality (4.5) and suppressing the summands, we write

$$\sum = \sum_{\mathrm{N}\mathfrak{p} \in (0,y] \cup (y^{c_{11}},\infty)} + \sum_{\mathfrak{a} \text{ not prime}} + \sum_{y < \mathrm{N}\mathfrak{p} \leq y^{c_{11}}}.$$

We will estimate these three sums separately.

We use Lemma 3.5 and (4.3) to obtain

$$\begin{aligned}\Big|\eta \sum_{\mathrm{N}\mathfrak{p}\in(0,y]\cup(y^{c_{11}},\infty)}\Big| &\ll \eta(110)^{-k}\Big(\sum_{\substack{\mathrm{N}\mathfrak{a}\leq y\\(\mathfrak{a},S)=1}}\frac{|\Lambda_{\pi\otimes\pi'}(\mathfrak{a})|}{\mathrm{N}\mathfrak{a}}+\sum_{(\mathfrak{a},S)=1}\frac{|\Lambda_{\pi\otimes\pi'}(\mathfrak{a})|}{\mathrm{N}\mathfrak{a}^{1+\eta/2}}\Big)\\ &\ll \eta(110)^{-k}(\eta^{-1}+\log y+d'd\log(\mathfrak{q}(\pi)\mathfrak{q}(\pi')))\\ &\ll (110)^{-k}(1+\eta\log y+\eta\mathcal{L})\ll k(110)^{-k}.\end{aligned}$$

Since $\eta\leq 1/55$, the identity $\sum_{m\geq 0}j_m(u)=1$ implies that

$$\mathrm{N}\mathfrak{a}^{-1/2}j_k(\eta\log(\mathrm{N}\mathfrak{a}))=(2\eta)^k\mathrm{N}\mathfrak{a}^{-\eta}j_k(\log(\mathrm{N}\mathfrak{a})/2)\leq(110)^{-k}\mathrm{N}\mathfrak{a}^{-\eta}.$$

Thus, as above,

$$\Big|\eta\sum_{\mathfrak{a}\text{ not prime}}\Big|\ll\eta(110)^{-k}\sum_{\substack{\mathfrak{a}=\mathfrak{p}^m\\m\geq 2\\(\mathfrak{a},S)=1}}\frac{|\Lambda_{\pi\otimes\pi'}(\mathfrak{a})|}{\mathrm{N}\mathfrak{a}^{1/2+\eta}}\ll\eta(110)^{-k}\sum_{(\mathfrak{a},S)=1}\frac{|\Lambda_{\pi\otimes\pi'}(\mathfrak{a})|}{\mathrm{N}\mathfrak{a}^{1+2\eta}}\ll k(110)^{-k}.$$

as well. Finally, recall that

$$S_{y,u}(\tau,\pi\otimes\pi')=\sum_{y<\mathrm{N}\mathfrak{p}\leq u}\frac{\Lambda_{\pi\otimes\pi'}(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^{1+i\tau}}.$$

Summation by parts gives us

$$\sum_{y<\mathrm{N}\mathfrak{p}\leq y^{c_{11}}}=S_{y,y^{c_{11}}}(\tau,\pi\otimes\pi')j_k(\eta\log y^{c_{11}})-\eta\int_y^{y^{c_{11}}}S_{y,u}(\tau,\pi\otimes\pi')j_k'(\eta\log u)\frac{du}{u}$$

since $S_{y,y}(\tau,\pi\otimes\pi')=0$. Much like the above calculations,

$$|\eta S_{y,y^{c_{11}}}(\tau,\pi\otimes\pi')j_k(\eta\log y^{c_{11}})|\ll\eta(110)^{-k}y^{-c_{11}\eta/2}\sum_{\substack{\mathrm{N}\mathfrak{p}\leq y^{c_{11}}\\(\mathfrak{p},S)=1}}\frac{|\Lambda_{\pi\otimes\pi'}(\mathfrak{p})|}{\mathrm{N}\mathfrak{p}}\ll k(110)^{-k}.$$

Therefore, since $|j_k'(u)|=|j_{k-1}(u)-j_k(u)|\leq j_{k-1}(u)+j_k(u)\leq 1$, we have

$$\Big|\eta\sum_{y<\mathrm{N}\mathfrak{p}\leq y^{c_{11}}}\Big|\leq\eta^2\int_y^{y^{c_{11}}}|S_{y,u}(\tau,\pi\otimes\pi')|\frac{du}{u}+O(k(110)^{-k}).$$

However, by (4.3) and the bound $\eta\gg\mathcal{L}^{-1}$, it follows that if k is sufficiently large, then each term of size $O(k(110)^{-k})$ is at most $\frac{1}{16}(100)^{-(k+1)}$. The lemma follows. □

4.3. **Zero detection: The proof of Proposition 4.1.** We now combine our upper and lower bounds on the derivatives of F to prove Proposition 4.1. Thus, we wish to show that if ρ_0 is a zero satisfying $|\rho_0-(1+i\tau)|\leq\eta$ and $\mathrm{Re}(\rho)\leq 1-c_8/\mathcal{L}$, then

$$\frac{y^{c_{10}\eta}}{(\log y)^3}\int_y^{y^{c_{11}}}|S_{y,u}(\tau,\pi\otimes\pi')|^2\frac{du}{u}\gg 1.$$

Combining Lemmas 4.2 and 4.4, we find that if if $|\rho_0-(1+i\tau)|\leq\eta$ and $\mathrm{Re}(\rho)\leq 1-c_8/\mathcal{L}$, then

$$\eta^2\int_y^{y^{c_{11}}}|S_{y,u}(\tau,\pi\otimes\pi')|\frac{du}{u}\geq\frac{1}{4}(100)^{-(k+1)}.$$

Using (4.3), we have

$$\eta^2 \int_y^{y^{c_{11}}} |S_{y,u}(\tau, \pi\otimes\pi')|\frac{du}{u} \gg y^{-c_{10}\eta/4},$$

where c_{10} is sufficiently large. Multiplying both sides by $y^{-c_{10}\eta/4}$ yields

$$y^{-c_{10}\eta/4}\eta^2 \int_y^{y^{c_{11}}} |S_{y,u}(\tau, \pi\otimes\pi')|\frac{du}{u} \gg y^{-c_{10}\eta/2}.$$

Using (4.3) again, we have that $y^{-c_{10}\eta/4}\eta^2 \ll (\log y)^{-2}$, so

$$\frac{1}{(\log y)^2} \int_y^{y^{c_{11}}} |S_{y,u}(\tau, \pi\otimes\pi')|\frac{du}{u} \gg y^{-c_{10}\eta/2}.$$

Squaring both sides and applying the Cauchy-Schwarz inequality yields the proposition.

4.4. **Proof of Theorem 1.5.** We conclude this section with the proof of Theorem 1.5. It suffices to consider the case where $d, d' \neq 1$ since these cases are handled by Theorem 1.1. If $d = d' = 2$, then by work of Ramakrishnan [45, Theorem M], $\pi\otimes\pi'$ is an isobaric sum of cuspidal automorphic representations of degree at most 4. Thus $L(s, \pi\otimes\pi', K)$ factors as a product of L-functions satisfying the hypotheses of Corollaries 1.2 and 1.4; the theorem follows upon applying the corollaries to each factor. If $d = 2$ and $d' = 3$, then by work of Kim and Shahidi [29] and Ramakrishnan and Wang [47], $\pi\otimes\pi'$ is an isobaric sum of cuspidal automorphic representations of degree at most 6. Thus $L(s, \pi\otimes\pi', K)$ also factors as a product of L-functions satisfying the hypotheses of Corollaries 1.2 and 1.4; the theorem follows upon applying the corollaries to each factor.

5. Proof of Theorems 1.6, 1.8, and 1.9

In this section, we consider the arithmetic applications of the zero-density estimates provided in Theorem 1.1 and Corollary 1.2 to approximate versions of Hoheisel's short interval prime number theorem. We prove Theorems 1.6 and 1.8, and Theorem 1.9 follows readily from Theorem 1.8 after addressing the issue of Landau-Siegel zeros.

5.1. **Proof of Theorem 1.6.** We require explicit formulae, such as (5.7), in order to study the right hand side of Theorem 1.6 without making reference to the size of $\Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{a})$. Note that the analogous result in [1, Equation 5.2] requires that the mean value of $\Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{a})$ remain bounded over short intervals (a straightforward consequence of GRC), and the analogous result in [42, Proof of Theorem 1.1] requires an asymptotic estimate for a certain sum of Dirichlet coefficients with a power-saving error term. Our explicit formula only uses the well-known fact that $\Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{a}) \geq 0$ for all $\mathfrak{a}$; it holds regardless of whether π is self-dual.

Let $x \geq 2$. Define

$$\psi_{\pi\otimes\tilde{\pi}}(x) = \int_0^x \Big(\sum_{\mathrm{N}\mathfrak{a}\leq t} \Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{a})\Big)dt = -\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{L'}{L}(s, \pi\otimes\tilde{\pi}, K)\frac{x^{s+1}}{s(s+1)}ds.$$

Note that the sum in the first integrand is monotonically increasing. Thus if $1 < y < x$, then by the mean value theorem,

$$-\frac{\psi_{\pi\otimes\tilde{\pi}}(x-y) - \psi_{\pi\otimes\tilde{\pi}}(x)}{y} \leq \sum_{\mathrm{N}\mathfrak{a}\leq x} \Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{a}) \leq \frac{\psi_{\pi\otimes\tilde{\pi}}(x+y) - \psi_{\pi\otimes\tilde{\pi}}(x)}{y}. \tag{5.1}$$

By a standard residue theorem computation,

$$\psi_{\pi\otimes\tilde{\pi}}(x)=\frac{x^2}{2}-\sum_{\rho\neq 0,-1}\frac{x^{\rho+1}}{\rho(\rho+1)}-(\mathrm{Res}_{s=0}+\mathrm{Res}_{s=-1})\frac{L'}{L}(s,\pi\otimes\tilde{\pi},K)\frac{x^{s+1}}{s(s+1)},$$

where ρ runs over all zeros of $L(s,\pi\otimes\tilde{\pi},K)$. Thus

$$\begin{aligned}\pm\frac{\psi_{\pi\otimes\tilde{\pi}}(x\pm y)-\psi_{\pi\otimes\tilde{\pi}}(x)}{y}&=x\pm\frac{y}{2}\mp\sum_{\rho\neq 0,-1}\frac{x^{\rho+1}((1\pm\frac{y}{x})^{\rho+1}-1)}{y\rho(\rho+1)}\\&\mp(\mathrm{Res}_{s=0}+\mathrm{Res}_{s=-1})\frac{L'}{L}(s,\pi\otimes\tilde{\pi},K)\frac{x^{s+1}((1\pm\frac{y}{x})^{s+1}-1)}{ys(s+1)}.\end{aligned}\tag{5.2}$$

We first address the sum over zeros $\rho=\beta+i\gamma$, restricting our attention to those ρ for which $0<\beta<1$. Observe that for each such ρ,

$$\mp\frac{x^{\rho+1}((1\pm\frac{y}{x})^{\rho+1}-1)}{y\rho(\rho+1)}=-\frac{x^\rho}{\rho}\mp yw_\rho^{\pm}x^{\rho-1},\tag{5.3}$$

where

$$w_\rho^{\pm}:=\frac{(1\pm\frac{y}{x})^{\rho+1}-1\mp(\rho+1)\frac{y}{x}}{\rho(\rho+1)(\frac{y}{x})^2}.$$

Since $0<\beta<1$ and $1<y<x$, a minor change in the proof of [19, Lemma 2.1] yields the bound $|w_\rho^{\pm}|\leq 1$ in both $\pm$ cases. Thus for any $1\leq T\leq x$, the sum over zeros $\rho=\beta+i\gamma$ with $0<\beta<1$ in (5.2) equals

$$-\sum_{\substack{|\gamma|\leq T\\0<\beta<1}}\frac{x^\rho}{\rho}+O\Big(\sum_{\substack{|\gamma|\geq T\\0<\beta<1}}\Big|\frac{x^{\rho+1}((1\pm\frac{y}{x})^{\rho+1}-1)}{y\rho(\rho+1)}\Big|+y\sum_{\substack{|\gamma|\leq T\\0<\beta<1}}x^{\beta-1}\Big).\tag{5.4}$$

Using (4.1) and the fact that $1<y<x$, we see that the first sum over zeros in the error term of (5.4) is

$$\ll\frac{x^2}{y}\sum_{\substack{|\gamma|\geq T\\0<\beta<1}}\frac{1}{|\rho|^2}\ll[K:\mathbb{Q}]d^2\frac{x^2(\log T)\log\mathfrak{q}(\pi)}{yT}.$$

We choose $x\geq\mathfrak{q}(\pi)^{d^2}$ and

$$y=\frac{x(\log x)^3}{\sqrt{T}}\Big(\sum_{\substack{|\gamma|\leq T\\0<\beta<1}}x^{\beta-1}\Big)^{-1/2}\tag{5.5}$$

so that the sum over zeros $\rho=\beta+i\gamma$ in (5.2) equals

$$-\sum_{\substack{|\gamma|\leq T\\0<\beta<1}}\frac{x^\rho}{\rho}+O\Big(\frac{x(\log x)^3}{\sqrt{T}}\Big(\sum_{\substack{|\gamma|\leq T\\0<\beta<1}}x^{\beta-1}\Big)^{1/2}\Big).\tag{5.6}$$

With our choice of y, the contribution to the sum over zeros $\rho=\beta+i\gamma$ in (5.2) with $\beta\leq 0$ is smaller than the error term in (5.6). The same can be said for the contribution from the residues in (5.2), which can be bounded using [25, Equation 5.24]. Collecting all of our estimates, we now see that

$$\sum_{\mathrm{N}\mathfrak{a}\le x}\Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{a})=x-\sum_{\substack{|\gamma|\le T\\ 0<\beta<1}}\frac{x^{\rho}}{\rho}+O\Big(\frac{x(\log x)^3}{\sqrt{T}}\Big(\sum_{\substack{|\gamma|\le T\\ 0<\beta<1}}x^{\beta-1}\Big)^{1/2}\Big). \tag{5.7}$$

Therefore, for any $1\le h\le x$,

$$\begin{aligned}\Big|\sum_{x<\mathrm{N}\mathfrak{a}\le x+h}\Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{a})-h\Big|&=\Big|-\sum_{\substack{|\gamma|\le T\\ 0<\beta<1}}\frac{(x+h)^{\rho}-x^{\rho}}{\rho}+O\Big(\frac{x(\log x)^3}{\sqrt{T}}\Big(\sum_{\substack{|\gamma|\le T\\ 0<\beta<1}}x^{\beta-1}\Big)^{1/2}\Big)\Big|\\ &\le h\sum_{\substack{|\gamma|\le T\\ 0<\beta<1}}x^{\beta-1}+O\Big(\frac{x(\log x)^3}{\sqrt{T}}\Big(\sum_{\substack{|\gamma|\le T\\ 0<\beta<1}}x^{\beta-1}\Big)^{1/2}\Big)\\ &=h\sum_{\substack{|\gamma|\le T\\ 0<\beta<1-c_8/\mathcal{L}}}x^{\beta-1}+O\Big(\frac{x(\log x)^3}{\sqrt{T}}\Big(\sum_{\substack{|\gamma|\le T\\ 0<\beta<1-c_8/\mathcal{L}}}x^{\beta-1}\Big)^{1/2}\Big)+o(h). \qquad (5.8)\end{aligned}$$

The $o(h)$ term arises from the at most four zeros $\beta+i\gamma$ with $1-c_8\mathcal{L}<\beta<1$ in Lemma 3.2.

To bound the sums over zeros in (5.8), note that by the functional equation for $L(s,\pi\otimes\tilde{\pi},K)$ and the zero-free region in Lemma 3.2 that

$$\sum_{\substack{|\gamma|\le T\\ 0<\beta<1-c_8/\mathcal{L}}}x^{\beta-1}\le 2\int_{1/2}^{1-c_8/\mathcal{L}}x^{\sigma-1}dN_{\pi\otimes\tilde{\pi}}(\sigma,T).$$

We let $A=4c_1d^2$, $x\ge([K:\mathbb{Q}]^{[K:\mathbb{Q}]}\mathfrak{q}(\pi)^2)^A$, and $T=\mathfrak{q}(\pi)^{-2/[K:\mathbb{Q}]}x^{1/(A[K:\mathbb{Q}])}\ge[K:\mathbb{Q}]$. Observe that if π satisfies GRC, then Theorem 1.1 and our choice of x and T imply that

$$\mathcal{L}\le\frac{1}{4c_1}\log x,\qquad N_{\pi\otimes\tilde{\pi}}(\sigma,T)\ll d^2(\mathfrak{q}(\pi)^2T^{[K:\mathbb{Q}]})^{c_1d^2(1-\sigma)}=d^2x^{\frac{1}{4}(1-\sigma)}. \tag{5.9}$$

Thus

$$\begin{aligned}\int_{1/2}^{1-c_8/\mathcal{L}}x^{\sigma-1}\,dN_{\pi\otimes\tilde{\pi}}(\sigma,T)&=x^{-1/2}N_{\pi\otimes\tilde{\pi}}(1/2,T)+\log x\int_{1/2}^{1-c_8/\mathcal{L}}x^{\sigma-1}N_{\pi\otimes\tilde{\pi}}(\sigma,T)\,d\sigma\\ &\ll x^{-3/8}+d^2\log x\int_{1/2}^{1-4c_1c_8/\log x}x^{\frac{3}{4}(\sigma-1)}\,d\sigma\\ &\ll x^{-3/8}+d^2(x^{-3/8}+e^{-3c_1c_8}). \qquad (5.10)\end{aligned}$$

If $L(s,\pi\otimes\pi',K)$ factors into a product of L-functions satisfying the hypotheses of Theorem 1.1 or Corollary 1.2, then we apply Theorem 1.1 and/or Corollary 1.2 to the factors of $L(s,\pi\otimes\pi',K)$ and deduce (5.10). (This can always be accomplished when $d=2$.)

Applying (5.10) to bound the sum over zeros in (5.8), it follows from the relationship between x and T that

$$\Big|\sum_{x<\mathrm{N}\mathfrak{a}\le x+h}\Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{a})-h\Big|\le c_{13}d^2e^{-3c_1c_8}h+o(h)+O(\mathfrak{q}(\pi)^{1/[K:\mathbb{Q}]}x^{1-\frac{1}{2A[K:\mathbb{Q}]}}(\log x)^3)$$

where c_{13} is sufficiently large. Because c_1 is both large and absolute, we may replace c_1 with the larger constant $\max\{c_1, (3c_8)^{-1}\log(4c_{13}d^2)\}$. This yields

$$\Big|\sum_{x<\mathrm{N}\mathfrak{a}\leq x+h}\Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{a})-h\Big|\leq\frac{h}{4}+o(h)+O(\mathfrak{q}(\pi)^{1/[K:\mathbb{Q}]}x^{1-\frac{1}{2A[K:\mathbb{Q}]}}(\log x)^3).$$

Finally, taking $h\gg\mathfrak{q}(\pi)^{1/[K:\mathbb{Q}]}x^{1-\frac{1}{2A[K:\mathbb{Q}]}}(\log x)^4$, the theorem follows when x is sufficiently large.

5.2. **An intermediate lemma.** For the remaining proofs, we require an analogue of (5.8) that holds for those $L(s,\pi,K)$ whose Dirichlet coefficients are not nonnegative. For convenience, we restrict our consideration to sums over primes.

Lemma 5.1. *Let $\pi\in\mathcal{A}_d(K)$. If $A=4c_1d$, $x\geq([K:\mathbb{Q}]^{[K:\mathbb{Q}]}\mathfrak{q}(\pi))^A$, $\sqrt{x}\leq h\leq x$, and $T=\mathfrak{q}(\pi)^{-2/[K:\mathbb{Q}]}x^{1/(A[K:\mathbb{Q}])}$, then*

$$\Big|\sum_{x<\mathrm{N}\mathfrak{p}\leq x+h}\Lambda_{\pi}(\mathfrak{p})-r(\pi)h\Big|\leq h\sum_{\substack{|\gamma|\leq T\\ 0<\beta<1}}x^{\beta-1}+O\Big(\frac{x(\log x)^5}{\sqrt{T}}\Big),$$

where $\beta+i\gamma$ runs through the zeros of $L(s,\pi,K)$.

Proof. For $1\leq y\leq x$, define

$$\phi(t)=\begin{cases}\min\{1,1+\frac{x-t}{y}\} & \text{if } 0\leq t\leq x+y,\\ 0 & \text{otherwise.}\end{cases}\tag{5.11}$$

If $\mathrm{Re}(s)>0$, then the Mellin transform of $\phi(t)$ at s is given by

$$\hat{\phi}(s)=\int_0^\infty\phi(t)t^{s-1}dt=\frac{x^{s+1}((1+\frac{y}{x})^{s+1}-1)}{ys(s+1)},$$

which is the same as (5.3) upon replacing s with ρ. For $T\geq1$, we see that

$$\sum_{\substack{\mathrm{N}\mathfrak{a}\leq x\\ (\mathfrak{a},S)=1}}\Lambda_\pi(\mathfrak{a})=-\int_{2-i\infty}^{2+i\infty}\frac{(L^S)'}{L^S}(s,\pi,K)\hat{\phi}(s)ds+O\Big(\sum_{\substack{x<\mathrm{N}\mathfrak{a}\leq x+y\\ (\mathfrak{a},S)=1}}|\Lambda_\pi(\mathfrak{a})|\Big).\tag{5.12}$$

Pushing the contour to the left, we see that the integral in (5.12) is equal to the expression in (5.2) with π replacing $\pi\otimes\tilde{\pi}$, where the $\rho=\beta+i\gamma$ now run through the zeros of $L^S(s,\pi,K)$. If we choose y as in (5.5), then the integral is estimated just as in the proof of Theorem 1.6. The most important difference is the fact that $\frac{(L^S)'}{L^S}(s,\pi,K)$ has poles at $(\log\alpha_\pi(j,\mathfrak{p}))/\log\mathrm{N}\mathfrak{p}$ for each $\mathfrak{p}\in S$. The real part of each of these poles is less than $1/2$ by (2.1).

It remains to bound the error term in (5.12) and show that the difference between the sum over integral ideals $\mathfrak{a}$ in (5.12) differs negligibly from the sum over primes $\mathfrak{p}$ in the statement of the lemma; once these tasks are completed, the proof proceeds much like the proof of Theorem 1.6. We accomplish both of these tasks by using Lemma 3.5 and the identity $\Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{a})=|\Lambda_\pi(\mathfrak{a})|^2$ for $(\mathfrak{a},S)=1$. We see that for our choice of x,

$$\sum_{\substack{\mathrm{N}\mathfrak{a}\leq x\\ (\mathfrak{a},S)=1}}\Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{a})\leq x\sum_{\substack{\mathrm{N}\mathfrak{a}\leq x\\ (\mathfrak{a},S)=1}}\frac{\Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{a})}{\mathrm{N}\mathfrak{a}}\ll x\log x.$$

The Cauchy-Schwarz inequality then yields the bounds

$$\Big|\sum_{\substack{\mathrm{N}\mathfrak{a}\le x\\ (\mathfrak{a},S)=1}}\Lambda_\pi(\mathfrak{a})-\sum_{\mathrm{N}\mathfrak{p}\le x}\Lambda_\pi(\mathfrak{p})\Big|\ll\Big(\sum_{\substack{\mathrm{N}\mathfrak{p}^m\le x\\ m\ge 2\\ \mathfrak{p}\notin S}}\log\mathrm{N}\mathfrak{p}\Big)^{1/2}\Big(\sum_{\substack{\mathrm{N}\mathfrak{p}^m\le x\\ m\ge 2\\ \mathfrak{p}\notin S}}\Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{a})\Big)^{1/2}+x^{1/2}(\log x)\sum_{\mathfrak{p}\in S}1$$

(which is $\ll\sqrt{x}(\log x)^2$) and, for $1\le y\le x$,

$$\sum_{x<\mathrm{N}\mathfrak{a}\le x+y}|\Lambda_\pi(\mathfrak{a})|\ll\Big(\sum_{x<\mathrm{N}\mathfrak{p}^m\le x+y}\Lambda_{\pi\otimes\tilde{\pi}}(\mathfrak{p}^m)\Big)^{1/2}\Big(\sum_{x<\mathrm{N}\mathfrak{p}^m\le x+y}\log\mathrm{N}\mathfrak{p}\Big)^{1/2}\ll[K:\mathbb{Q}]\sqrt{xy\log x}.$$

(For the second bound, we used the Brun-Titchmarsh theorem [37] and the fact that at most $[K:\mathbb{Q}]$ prime ideals of K lie over a given rational prime.)

□

5.3. **The Sato-Tate conjecture.** Following Shahidi [50, pg. 162], we now specify the representations π for which we expect the Sato-Tate conjecture to hold. Let K be a totally real field, and let $\pi\in\mathcal{A}_2(K)$ be non-CM and have trivial central character. There are two types of exceptional representations we would like to exclude: monomial representations, and representations of Galois type. A representation π is a *monomial representation* if there exists a nontrivial character χ of $K^\times\setminus\mathbb{A}_K^\times$ such that $\pi\otimes\chi\cong\pi$. A representation ρ is of *Galois type* if for every archimedean place $\mathfrak{v}$ of K, the Langlands parameters $\rho_{\mathfrak{v}}$ (which are representations of the Weil group of $K_{\mathfrak{v}}$) associated with the archimedean components $\pi_{\mathfrak{v}}$ factor through the Galois group of $\bar{K}_{\mathfrak{v}}/K_{\mathfrak{v}}$. We say that π is **genuine** if it is neither monomial nor of Galois type.

Examples of genuine π include those associated to non-CM Hilbert modular forms over totally real number fields with each weight both even and at least 2 (including non-CM newforms over $\mathbb{Q}$ of even weight $k\ge 2$) and Hecke-Maass forms whose Laplace eigenvalue is not equal to 1/4. As elliptic curves over real quadratic fields are known to be modular [14], our results apply unconditionally to the representations π associated to such curves.

Recall that the Sato-Tate conjecture concerns the distribution of the quantities $\lambda_\pi(\mathfrak{p})=2\cos\theta_{\mathfrak{p}}$ as $\mathfrak{p}$ ranges over primes for which $\pi_{\mathfrak{p}}$ is unramified, where $\theta_{\mathfrak{p}}\in[0,\pi]$. At each such prime $\mathfrak{p}$, the local factor of the n-th symmetric power L-function is given by

$$L_{\mathfrak{p}}(s,\mathrm{Sym}^n\pi,K)=\prod_{j=0}^{n}(1-e^{i\theta_{\mathfrak{p}}(n-2j)}\mathrm{N}\mathfrak{p}^{-s})^{-1}=\sum_{k=0}^{\infty}\frac{U_n(\cos(k\theta_{\mathfrak{p}}))}{\mathrm{N}\mathfrak{p}^s},$$

where U_n is the n-th Chebyshev polynomial of the second kind. At ramified primes $\mathfrak{p}$, it follows from [43] there are numbers $\beta_{\mathrm{Sym}^n\pi}(j,\mathfrak{p})$ of absolute value at most $\mathrm{N}\mathfrak{p}^{\frac{1}{2}-\frac{1}{(n+1)^2+1}}$ for which the local factor is given by

$$L_{\mathfrak{p}}(s,\mathrm{Sym}^n\pi,K)=\prod_{j=0}^{n}(1-\beta_{\mathrm{Sym}^n\pi}(j,\mathfrak{p})\mathrm{N}\mathfrak{p}^{-s})^{-1}.$$

(If $\mathfrak{p}$ is ramified, then some of the $\beta_{\mathrm{Sym}^n\pi}(j,\mathfrak{p})$ might equal zero.) We note that $L(s,\mathrm{Sym}^1\pi,K)=L(s,\pi,K)$ and $L(s,\mathrm{Sym}^0\pi,K)=\zeta_K(s)$.

In Theorem 1.8, our goal is to estimate for $I\subseteq[-1,1]$ the summation

$$\sum_{\substack{x<\mathrm{N}\mathfrak{p}\le x+h\\ \mathfrak{p}\notin S}}\mathbf{1}_I(\cos\theta_{\mathfrak{p}})\log\mathrm{N}\mathfrak{p}\tag{5.13}$$

where S is the set of $\mathfrak{p}$ for which π is ramified and $h \geq x^{1-\delta}$ for some $\delta > 0$. Recall from the discussion before Theorem 1.8 in Section 1 that I can be Sym^N-minorized if there exist $b_0, \ldots, b_N \in \mathbb{R}$ such that $b_0 > 0$ and (1.10) holds for all $t \in [-1,1]$. Thus, if I can be Sym^n-minorized, we can obtain a non-trivial lower bound for (5.13) by considering an appropriate linear combination of the logarithmic derivatives of $L(s, \mathrm{Sym}^n\pi, K)$ for $n \leq N$.

Proof of Theorem 1.8. Let S be the set of prime ideals $\mathfrak{p}$ for which $\pi_{\mathfrak{p}}$ is ramified. The upper bound follows from the Brun-Titchmarsh theorem [37], so we proceed to the lower bound. Suppose that $I \subset [-1,1]$ can be Sym^n-minorized and that $L(s, \mathrm{Sym}^n\pi, K)$ is automorphic for each $0 \leq n \leq N$. Let $b_0, \ldots, b_n$ be as in (1.10) and set $B = \max_{0\leq n\leq N} |b_n|/b_0$. Let $A = 4c_1(N+1)$,

$$x \geq ([K:\mathbb{Q}]^{[K:\mathbb{Q}]} \max_{0\leq n\leq N} \mathfrak{q}(\mathrm{Sym}^n\pi))^A,$$

and $T = \mathfrak{q}(\mathrm{Sym}^n\pi)^{-1/[K:\mathbb{Q}]} x^{1/(A[K:\mathbb{Q}])}$.

First, observe that x is larger than the norm of any $\mathfrak{p} \in S$. Thus

$$\sum_{x < \mathrm{N}\mathfrak{p} \leq x+h} U_n(\cos\theta_{\mathfrak{p}}) \log \mathrm{N}\mathfrak{p} = \sum_{x < \mathrm{N}\mathfrak{p} \leq x+h} \Lambda_{\mathrm{Sym}^n\pi}(\mathfrak{p}). \tag{5.14}$$

This establishes the lower bound

$$\sum_{x < \mathrm{N}\mathfrak{p} \leq x+h} \mathbf{1}_I(\cos\theta_{\mathfrak{p}}) \log \mathrm{N}\mathfrak{p} \geq \sum_{n=0}^{N} b_n \sum_{x < \mathrm{N}\mathfrak{p} \leq x+h} \Lambda_{\mathrm{Sym}^n\pi}(\mathfrak{p}). \tag{5.15}$$

By Lemma 5.1 and a calculation nearly identical to (5.10), we deduce the existence of a sufficiently large $c_{14} > 0$ such that

$$\sum_{x < \mathrm{N}\mathfrak{p} \leq x+h} \mathbf{1}_I(\cos\theta_{\mathfrak{p}}) \log \mathrm{N}\mathfrak{p} \geq b_0\Big((1 - N(c_{14}Be^{-3c_1c_8} - o(1)))h - O(BNx^{1-\frac{1}{2A[K:\mathbb{Q}]}}(\log x)^5)\Big).$$

As before, we make c_1 sufficiently large, and the $o(1)$ term arises from the contributions of the at most four zeros present in Lemma 3.2. Because c_1 is large and absolute, we may replace c_1 with the larger constant $\max\{c_1, (3c_8)^{-1}\log(4c_{14}BN)\}$, so

$$\sum_{x < \mathrm{N}\mathfrak{p} \leq x+h} \mathbf{1}_I(\cos\theta_{\mathfrak{p}}) \log \mathrm{N}\mathfrak{p} \geq b_0\Big(\Big(\frac{1}{4} - o(1)\Big)h - O(BNx^{1-\frac{1}{2A[K:\mathbb{Q}]}}(\log x)^5)\Big).$$

Choosing $h \gg BNx^{1-\frac{1}{2A[K:\mathbb{Q}]}}(\log x)^6$, we obtain the lower bound

$$\sum_{x < \mathrm{N}\mathfrak{p} \leq x+h} \mathbf{1}_I(\cos\theta_{\mathfrak{p}}) \log \mathrm{N}\mathfrak{p} \gg b_0 h(1 - o(1)). \tag{5.16}$$

when x is sufficiently large. □

Proof of Theorem 1.9. First, we determine the integers $0 \leq n \leq N$ for which $L(s, \mathrm{Sym}^n\pi, K)$ has a Landau-Siegel zero. Recall from the second part of Lemma 3.3 that a Landau-Siegel zero of $L(s, \mathrm{Sym}^n\pi, K)$ is a simple real zero β such that

$$\beta \geq 1 - c_8((n+1)^4 \log(\mathfrak{q}(\mathrm{Sym}^n\pi)[K:\mathbb{Q}]^{[K:\mathbb{Q}]}))^{-1}.$$

In [22, 3, 46], it is shown that $L(s, \pi, K)$, $L(s, \mathrm{Sym}^2\pi, K)$, and $L(s, \mathrm{Sym}^4\pi, K)$, respectively, have no Landau-Siegel zeros. As an application of [22, Theorem B], if $n \geq 3$ and $\mathrm{Sym}^j\pi \in \mathcal{A}_{j+1}(K)$ for $j \in \{n-2, n, n+2\}$, then $L(s, \mathrm{Sym}^n\pi, K)$ has no Landau-Siegel zero, provided that c_8 is sufficiently small. We conclude that if $N \geq 3$ and $L(s, \mathrm{Sym}^n\pi, K)$ is automorphic over K for all $n \le N$, then the only $n \le N$ for which $L(s, \mathrm{Sym}^n\pi, K)$ might have a Landau-Siegel zero are $n \in \{0, N-1, N\}$. By hypothesis, $\zeta_K(s)$ has no Landau-Siegel zero, so $L(s, \mathrm{Sym}^n\pi, K)$ might have a Landau-Siegel zero if $n \in \{N-1, N\}$

If we repeat the proof of Theorem 1.8 with $h = x$ using the zero-free region in Lemma 3.3, then we find that

$$\sum_{x<\mathrm{N}\mathfrak{p}\le 2x} \mathbf{1}_I(\cos\theta_{\mathfrak{p}})\log \mathrm{N}\mathfrak{p} \ge b_0\Big(\Big(\frac{1}{4}-o(1)\Big)x - O(BNx^{1-\frac{1}{2A[K:\mathbb{Q}]}}(\log x)^3)\Big),$$

provided that c_1 is sufficiently large and

$$x \ge \Big([K:\mathbb{Q}]^{[K:\mathbb{Q}]}\max_{0\le n\le N}\mathfrak{q}(\mathrm{Sym}^n\pi)\Big)^{4(N+1)^4\max\{c_1,(3c_8)^{-1}\log(4c_{14}BN)\}}. \tag{5.17}$$

Here, the $o(1)$ term comes from the possible Landau-Siegel zeros associated to $\mathrm{Sym}^{N-1}\pi$ and $\mathrm{Sym}^N\pi$. By hypothesis, I can be Sym^N-minorized without admitting Landau-Siegel zeros; thus $b_n \le 0$ for each $n \in \{N-1, N\}$ such that $L(s, \mathrm{Sym}^n\pi, K)$ has a Landau-Siegel zero. Therefore, if $n \in \{N-1, N\}$ and $L(s, \mathrm{Sym}^n\pi, K)$ has a Landau-Siegel zero, then such a Landau-Siegel zero gives a positive contribution to the lower bound in (5.16), and we may discard this contribution. We conclude that there is an unramified $\mathfrak{p}$ such that $\cos\theta_{\mathfrak{p}} \in I$ and $\mathrm{N}\mathfrak{p} \le 2x$, where x is given by (**??**).

It remains to bound $\max_{0\le n\le N}\mathfrak{q}(\mathrm{Sym}^n\pi)$. Let $0 \le n \le N$. For each unramified $\mathfrak{p}$, consider the identity $L_{\mathfrak{p}}(s, \pi\otimes\mathrm{Sym}^{n-1}\pi, K) = L_{\mathfrak{p}}(s, \mathrm{Sym}^n\pi, K)L_{\mathfrak{p}}(s, \mathrm{Sym}^{n-2}\pi, K)$. Using (4.1) to relate the arithmetic conductor of each side, we conclude by induction on n that $\log q(\mathrm{Sym}^n\pi) \ll n^3\log q(\pi) \ll N^3\log q(\pi)$. (See also Rouse [48, Lemma 2.1]. His proof gives an implied constant depending on $[K:\mathbb{Q}]$, but this dependence is easily removed.) From the shape of $L_\infty(s, \mathrm{Sym}^n\pi, K)$ given by Moreno and Shahidi [40], it follows that

$$\log\Big(\prod_{j=1}^{(n+1)[K:\mathbb{Q}]}(|\kappa_{\mathrm{Sym}^n\pi}(j)|+3)\Big) \ll n\log\Big(n\prod_{j=1}^{2[K:\mathbb{Q}]}(|\kappa_\pi(j)|+3)\Big),$$

and the result follows. In the special case that π corresponds to a newform of $\mathbb{Q}$ of squarefree level and trivial nebentypus, Cogdell and Michel [10] use the local Langlands correspondence to show that $\log q(\mathrm{Sym}^n\pi) = n\log q(\pi)$, which accounts for the claimed improvement. □

Remark. Suppose now that $\zeta_K(s)$ does have a Landau-Siegel zero. A slight reformulation of the proof of Theorem 1.8 with $h = x$ yields the lower bound

$$\sum_{x<\mathrm{N}\mathfrak{p}\le 2x}\mathbf{1}_I(\cos\theta_{\mathfrak{p}})\log\mathrm{N}\mathfrak{p} \ge b_0\Big(\sum_{x<\mathrm{N}\mathfrak{p}\le 2x}\log\mathrm{N}\mathfrak{p} - c_{14}BNe^{-3c_1c_8}x - O(BNx^{1-\frac{1}{2A[K:\mathbb{Q}]}}(\log x)^3)\Big),$$

Using the unconditional lower bound for $\sum_{x<\mathrm{N}\mathfrak{p}\le 2x}\log\mathrm{N}\mathfrak{p}$ that follows from [52, Theorem 5.2], we conclude that there exists a sufficiently large constant c_{15} such that

$$\sum_{x<\mathrm{N}\mathfrak{p}\le 2x}\mathbf{1}_I(\cos\theta_{\mathfrak{p}})\log\mathrm{N}\mathfrak{p} \ge b_0\Big(\frac{1}{([K:\mathbb{Q}]^{[K:\mathbb{Q}]}D_K)^{c_{15}}}x - c_{15}BNe^{-3c_1c_8}x - O(BNx^{1-\frac{1}{2A[K:\mathbb{Q}]}}(\log x)^3)\Big).$$

One can now easily find an effective value of x (which is at least as large as the upper bound in Theorem 1.8) such that there exists an unramified $\mathfrak{p}$ such that $\cos\theta_{\mathfrak{p}} \in I$ and $\mathrm{N}\mathfrak{p} \le 2x$. Here, c_1 needs to be sufficiently large with respect to B, K, and N.

6. Proof of Theorem 1.7

Let $\pi \in \mathcal{A}_2(\mathbb{Q})$ have trivial central character. Thus $\pi \ncong \pi \otimes \chi$ for all nontrivial primitive Dirichlet characters χ; in particular, π is not monomial. Throughout this section, χ will denote a primitive Dirichlet character of modulus q such that $(q, q(\pi)) = 1$. For such χ, we have that $\Lambda_{(\pi\otimes\chi)\otimes\tilde{\pi}}(n) = \Lambda_{\pi\otimes\tilde{\pi}}(n)\chi(n)$ for all $n \geq 1$. All implied constants in this section will depend at most on $\mathfrak{q}(\pi)$.

First, we address the possibility of a Landau-Siegel zero of $L(s, (\pi\otimes\chi)\otimes\tilde{\pi}, \mathbb{Q})$. We have the factorization $L(s, (\pi\otimes\chi)\otimes\tilde{\pi}, \mathbb{Q}) = L(s, \chi, \mathbb{Q})L(s, \mathrm{Sym}^2\pi\otimes\chi, \mathbb{Q})$. Banks [3] proved that $L(s, \mathrm{Sym}^2\pi\otimes\chi, \mathbb{Q})$ has no Landau-Siegel zero, and $L(s, \chi, \mathbb{Q})$ has no Laundau-Siegel zero except possibly in the case where χ is a real primitive character. Page proved that if c_8 is suitably small and $Q \geq 3$, then there is at most one real primitive Dirichlet character χ with modulus $q \leq Q$ for which $L(s, \chi, \mathbb{Q})$ has a Landau-Siegel zero β_1 satisfying $\beta_1 > 1 - c_8/\log Q$ [11, Chapter 14]. Therefore, as χ varies through the primitive Dirichlet characters modulo q with $q \leq Q$, we see that at most one of the L-functions $L(s, (\pi\otimes\chi)\otimes\tilde{\pi}, \mathbb{Q})$ has a Landau-Siegel zero β_1, and β_1 must also a Landau-Siegel zero of $L(s, \chi, \mathbb{Q})$.

Let $T \leq Q$, $x \leq hQ$ and $\log x \leq (\log Q)^2$. We apply Lemma 5.1 to each factor of $L(s, (\pi\otimes\chi)\otimes\tilde{\pi}, \mathbb{Q})$ and obtain

$$\sum_{x<n\leq x+h} \Lambda_{\pi\otimes\tilde{\pi}}(p)\chi(p) - \delta(\chi)h + h\xi^{\beta_1-1} \ll h\Big(\sum_{|\gamma|\leq T} x^{\beta-1} + \frac{Q^2}{\sqrt{T}}\Big),$$

for some $\xi \in [x, x+h]$, where the summation on the right-hand side is over the nontrivial zeros $\rho = \beta + i\gamma$ of $L(s, (\pi\otimes\chi)\otimes\tilde{\pi}, \mathbb{Q})$ which are not β_1. Since there are $O(Q^2)$ primitive Dirichlet characters to modulus $q \leq Q$, it follows that

$$\sum_{\substack{q\leq Q\\ \gcd(q,q(\pi))=1}} \sideset{}{^\star}\sum_{\chi \bmod q} \Big| \sum_{x<p\leq x+h} \Lambda_{\pi\otimes\tilde{\pi}}(p)\chi(p) - \delta(\chi)h + \delta_{q,*}(\chi)h\xi^{\beta_1-1}\Big|$$

$$\ll h\Big(\sum_{\substack{q\leq Q\\ \gcd(q,q(\pi))=1}} \sideset{}{^\star}\sum_{\chi \bmod q} \sum_{|\gamma|\leq T} x^{\beta-1} + \frac{Q^4}{\sqrt{T}}\Big). \tag{6.1}$$

If we let

$$\mathcal{N}(\sigma, Q, T) = \sum_{\substack{q\leq Q\\ \gcd(q,q(\pi))=1}} \sideset{}{^\star}\sum_{\chi \bmod q} N_{(\pi\otimes\chi)\otimes\tilde{\pi}}(\sigma, T),$$

then the triple sum in (6.1) is bounded by

$$\log x \int_{\frac{1}{2}}^{1} x^{\sigma-1}\mathcal{N}(\sigma, Q, T)d\sigma + x^{-1/2}\mathcal{N}(1/2, Q, T).$$

Using Theorem 1.5 and Lemma 3.3 and recalling our choice of T, we bound the above display by

$$\log x \int_{\frac{1}{2}}^{1-\frac{c_8}{\mathcal{L}'}} x^{\frac{1}{2}(\sigma-1)}d\sigma + x^{-\frac{1}{4}} \ll x^{-\frac{c_8}{2\mathcal{L}'}} + x^{-\frac{1}{4}},$$

where $\mathcal{L}' = 162\log(\mathfrak{q}(\pi)QT)$. If we choose $T = Q^{10}$, then the right-hand side of (6.1) is bounded as claimed in the statement of Theorem 1.7.

Appendix A. Sym^n-minorants

We close with two lemmas on Sym^n-minorants. The first explicitly classifies the intervals which can be Sym^4-minorized, i.e. those intervals in Theorem 1.8 we can access unconditionally for any $L(s,\pi,K)$. The second concerns the asymptotics of the smallest n needed to access the set of primes with $|\lambda_\pi(\mathfrak{p})| > 2(1-\delta)$ as $\delta \to 0$ and obtains an improvement over the naïve Fourier bound.

Lemma A.1. *Let* $\beta_0 = \frac{1+\sqrt{7}}{6} = 0.6076\ldots$ *and* $\beta_1 = \frac{-1+\sqrt{7}}{6} = 0.2742\ldots$. *The interval* $[a,b] \subseteq [-1,1]$ *can be* Sym^4*-minorized if and only if it satisfies one of the following conditions:*

(1) $a = -1$ *and* $b > -\beta_0$,
(2) $-1 < a \leq -\beta_0$ *and* $b > \frac{a+\sqrt{16a^4-11a^2+2}}{2(1-4a^2)}$,
(3) $-\beta_0 \leq a \leq -\beta_1$ *and* $b > \frac{-1}{6a}$,
(4) $-\beta_1 \leq a < \beta_1$ *and* $b > \frac{a+\sqrt{16a^4-11a^2+2}}{2(1-4a^2)}$, *and*
(5) $\beta_1 \leq a < \beta_0$ *and* $b = 1$.

Proof. We begin with sufficiency. For each case, we list a polynomial $F(x)$ which, for $x \in [-1,1]$, is positive only if $x \in [a,b]$. We then compute

$$b_0(F) := \int_{-1}^{1} F d\mu_{ST}$$

and verify that it is positive. This is sufficient, since any such $F(x)$ can be scaled to minorize the indicator function.

(1) $F(x) = (x-1)(x-b)(x-\beta_1)^2$ and $b_0(F) = (b+\beta_0)(\frac{14+\sqrt{7}}{36})$.
(2) $F(x) = -(x-a)(x-b)(x+\frac{a+b}{4ab+1})^2$ and $b_0(F) = \frac{(1-4a^2)b^2-ab+a^2-1/2}{4(4ab+1)}$.
(3) $F(x) = (x-1)(x+1)(x-a)(x-b)$ and $b_0(F) = -\frac{3}{4}(ab+\frac{1}{6})$.
(4) $F(x) = -(x-a)(x-b)(x+\frac{a+b}{4ab+1})^2$.
(5) $F(x) = (x+1)(x-a)(x+\beta_1)^2$ and $b_0(F) = (\beta_0-a)(\frac{14+\sqrt{7}}{36})$.

The proof of necessity necessarily involves tedious casework, which we omit. Let us say only that we consider polynomials $F(x)$, ordered by degree, the number of real roots, and the placement of those roots relative to $a, b, 1$, and -1, and in each case we determine conditions under which $b_0(F) > 0$. ∎

Lemma A.2. *If* $n \geq 1$, *then the set* $[-1,-a] \cup [a,1]$ *can be* Sym^{2n}*-minorized if* $a < \sqrt{1-\frac{3/2}{n+1}}$.

Proof. We recall the well-known fact that

$$\int_{-1}^{1} x^{2m}\, d\mu_{\mathrm{ST}} = \frac{1}{m+1}\binom{2m}{m} =: C_m.$$

Given n and a satisfying the conditions of the lemma, we use the minorant $f_{n,a}(x) = (x^2 - a^2)x^{2n-2}$, and we find that

$$\int_{-1}^{1} f_{n,a}\, d\mu_{\mathrm{ST}} = \frac{C_{n-1}}{4^{n-1}}\Big(1 - a^2 - \frac{3/2}{n+1}\Big).$$

∎

Remark. The Sato-Tate measures of the sets considered in Lemma A.2 satisfy $\mu^{-1} \gg n^{3/2}$, so the minorants in the proof provide a significant improvement over those arising from a naïve Fourier approximation.

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